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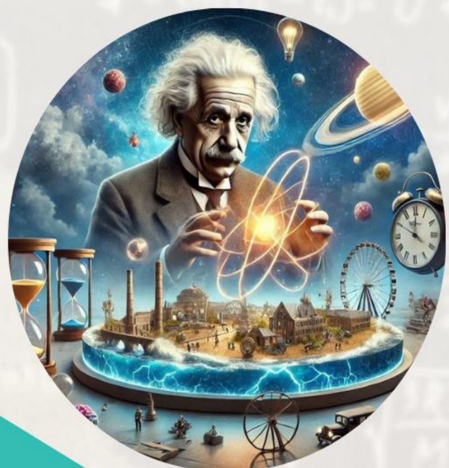

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Data Applications of Alpha Power Inverted Exponentiated Weibull Distribution

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Abstract: The Inverted Exponentiated Weibull (IEW) distribution is extended in a new way by using the generator Alpha Power (AP) transformation. Several sub-models are included into the final extended distribution, which is called the Alpha Power Inverted Exponentiated Weibull (APIEW) distribution. All the statistical properties of the new distribution, including the hazard rate, mean residual life, mean inactivity time, quantile, moments, Rényi entropy, and order statistics, are computed. The maximum likelihood estimation method is used to estimate the unknown parameters of the proposed distribution. Next, two sets of application data are used to show how the model may be adjusted.

Keywords: Inverted Exponentiated Weibull, Alpha Power Transformation, Hazard Rate Function, Moments, Rényi Entropy, Maximum Likelihood Estimation, Simulation

Introduction

Research on life dependability and testing makes extensive use of the Inverse Weibull (IW) distribution. According to (Drapella 1993; Mudholkar and Kollia 1994), it is thought of as the inverse of the traditional Weibull distribution. In diesel engines, this distribution is used to describe the wear and tear of mechanical elements, such as the crankshaft and pistons, as pointed out by Keller et al. (1982). Here are the equations that represent the IW distribution's Cumulative Distribution Function (CDF) and Probability Density Function (PDF):

$F(x) = e^{-vx^{-\eta}}, x \geq 0, v > 0, \eta > 0$ (1) And:

$f(x) = v\eta x^{-(\eta+1)} e^{-vx^{-\eta}}, x \geq 0, v > 0, \eta > 0$ (2)

where, v is the scale parameter and η is the shape parameter.

In recent years, various generalizations of the inverse Weibull distribution have been studied. For example, Pararai et al. (2014) examined the Kumaraswamy modified inverse Weibull distribution, Elbatal et al. (2015) studied the reflected generalized beta inverse Weibull distribution, Okasha et al. (2017) and Okasha et al. (2020a-b; 2021; 2022) studied the Marshall-Olkin extended inverse Weibull distribution, Lin et al. (2023) studied the Bayesian estimation of Marshall Olkin extended inverse Weibull under progressive type II censoring, and Saboori et al. (2020) examined the Generalized modified inverse Weibull distribution.

The focus is on the Inverted Exponentiated Weibull (IEW) distribution, as introduced by De Gusmão *et al*. (2012); and Lee *et al*. (2017) which is based on the transformation $Z = \frac{1}{X}$, where X follows the Exponentiated Weibull (EW) distribution. The Cumulative Distribution Function (CDF) and Probability Density Function (PDF) of the IEW distribution are provided accordingly:

$F(x) = 1 - (1 - e^{-vx^{-\eta}})^{\xi}, x \geq 0, v > 0, \eta > 0, \xi > 0$ (3) And:

various researchers. These include the generalized inverse

$-(\eta+1) \; -vx^{-\eta} \, ($

$_{-vx^{-\eta}}\big)^{\xi-1}$ (4) $f\ \eta\xi x e\big|$ 1

$-e$ | $,x \geq 0, v > 0, \eta > 0, \xi > 0$

Weibull distribution proposed by De Gusmão *et al.* (2011), the modified inverse Weibull distribution introduced by Khan and King (2012), the beta inverse Weibull model by Hanook *et al.* (2013), the gamma

where, *η* and *ξ* are the shape parameters and *v* is the scale parameter of IEW distribution. Bayesian parameter estimation of the IEW distribution has been previously examined by Lee *et al.* (2017).

$G_{APT}(x) = \begin{cases} \frac{v\eta\xi \log(\alpha)}{\alpha - 1} \end{cases}$ $1-e^{-vx^{-\eta}}$ α

$x^{-(\eta+1)} e^{-vx}$

$\left(\ \ ^{-\eta}\right)^{\xi-1} \ 1-(1-e^{-vx^{-\eta}})^{\xi}$

$,x \geq 0, \alpha > 0, \alpha \neq 1$ (8)

In contrast, (Mahdavi and Kundu, 2017) introduced a modification to the underlying Cumulative Distribution Function (CDF) by incorporating an additional parameter to generate a range of distributions. This approach is referred to as the Alpha Power Transformation (APT). If *F*(*x*) represents the CDF of any distribution, then the CDF and PDF of the APT can be expressed as:

$v\eta\xi x^{-(\eta+1)} e^{-vx^{-\eta}} \left(1-e^{-vx^{-\eta}}\right)^{\xi-1}, \alpha = 1$

where, $v > 0$, $\eta > 0$, $\xi > 0$.

Through the application of the generalized binomial expansion and the power series, a valuable linear portrayal of the Probability Density Function (PDF) is derived. (if $\alpha > 0$, $\alpha \neq 1$) as:

$$G_{APT}(x) = \begin{cases} \dfrac{\alpha^{F(x)} - 1}{\alpha - 1}, & \alpha > 0, \alpha \neq 1 \quad (5) \\ F(x), & \alpha = 1 \end{cases}$$

$$g_{APIEW}(x) = \sum_{m=0}^{\infty} W_m v\eta (m+1) x^{-(\eta+1)} e^{-(m+1)vx^{-\eta}} \quad (9)$$

where:

And: $W_m = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} (-1)^{j+m} \dfrac{\xi(\xi j + \xi - 1)! (\log(\alpha))^{k+1}}{j!(\xi j + \xi - m - 1)!(m+1)!k!(\alpha - 1)} \binom{k}{j}$

$$g_{APT}(x) = \begin{cases} \dfrac{Log(\alpha)}{\alpha - 1} f(x)\alpha^{F(x)}, & \alpha > 0, \alpha \neq 1 \\ f(x), & \alpha = 1 \end{cases} \quad (6)$$

Several sub-models of the APIEW distribution are enumerated in the Table (1).

Figure (1) provides a graphical illustration of the PDF corresponding to various parameter values.

The APT distribution has been extensively studied, with various distributions such as the alpha power Weibull distribution by Nassar *et al.* (2017; 2019), the alpha power Gompertz distribution by Eghwerido *et al.* (2021), the alpha power transformed inverse Lindley distribution by Dey *et al.* (2019), and alpha power inverse Weibull distribution by Basheer *et al.* (2019; 2021; 2022). The APIEW distribution is introduced in this study as a new modification of the IEW distribution with four parameters. It encompasses a range of lifetime distributions, including the inverse exponential, inverse Rayleigh, IW, alpha power IW, and IEW distributions, as special cases. The APIEW distribution is highlighted due to its inclusion of twelve-lifetime distributions as sub- models and its PDF representation as a mixture of IW distribution, which proves advantageous for deriving its

Reliability Analysis

The reliability function of APIEW distribution is defined as follows:

$\left[\ \dfrac{\alpha - \alpha^{1-\left(1-e^{-vt^{-\eta}}\right)^{\xi}}}{\alpha - 1}\ \right], t \geq 0, \alpha > 0, \alpha \neq 1$ (10)

$\left(1-e^{-vt^{-\eta}}\right)^{\xi}$,α =1

Hazard Rate Function

The HR function of APIEW distribution is defined as follows:

key properties.

$g(t)$

$\log(\alpha)v\eta\xi t^{-(\eta+1)}e^{-vt^{-\eta}}\left(1-e^{-vt^{-\eta}}\right)^{\xi-1}$,$t \geq 0, \alpha > 0, \alpha \neq 1$

$-vt^{-\eta}$ ξ

(11)

Alpha Power Inverted Exponentiated Weibull

$h(t) = \; = \; -1+\alpha\left(1-e\right)$

Distribution

By inserting the CDF of the IEW distribution given by $R(t)$

$\dfrac{v\eta\xi t^{-(\eta+1)}e^{-vt^{-\eta}}}{1-e^{-vt^{-\eta}}}$

,α =1

(3) in the CDF of the APT distribution given by (5), we get the CDF of a new distribution denoted as APIEW $(x;\alpha,v,\eta,\xi)$ distribution given by:The graphical representations of the HRF for various parameter values are illustrated in Fig. (2).

Reversed Hazard Rate Function

$G_{APT}(\alpha x) = \dfrac{1-\left(1-e^{-vx^{-\eta}}\right)^{\xi}}{\alpha-1}$

-1 , $x \geq 0, \alpha > 0, \alpha \neq 1$

(7)

The reversed hazard rate (RHR) function of APIEW distribution is defined as follows:

$1-\left(1-e\right.$

$\left.{}^{-vx^{-\eta}}\right)^{\xi}$

,α =1

$-(\eta+1)\; -vt^{-\eta} \; \left(e \right)$

$1-\alpha$

$_{\xi}\ ,t \geq 0, \alpha > 0, \alpha \neq 1$

$-1+1-e^{-vt}$

$$( ) = \frac{g(t)}{} = \log(\alpha)v\eta\xi t \quad e \quad -\frac{1-}{} \quad {}^{-vt^{-\eta}}\ {}^{\xi-1} \quad \left({}^{-\eta}_{t} \right)$$

(12) $-(\eta+1)\ {}^{-vt^{-\eta}}\ {}_{-vt^{-\eta}}\ {}^{\xi-1}$

Table 1: Sub-models of the APIEW (α, v, η, ξ) distribution

Models	Parameters			
Inverted Exponentiated Weibull (IEW)	1	ν	η	ξ
Inverted Exponentiated Fréchet (IEF)	1	1	η	ξ
Inverted Exponentiated Rayleigh (IER)	1	ν	2	ξ
Inverted Exponentiated Exponential (IEE)	1	ν	1	ξ
Alpha Power Inverse Weibull (APIW)	α	ν	η	1
Alpha Power Fréchet (APF)	α	1	η	1
Alpha Power Inverse Rayleigh (APIR)	α	ν	2	1
Alpha Power Inverse Exponential (APIE)	α	ν	1	1
Inverse Weibull (IW)	1	ν	η	1
Fréchet (F)	1	1	η	1
Inverse Rayleigh (IR)	1	ν	2	1
Inverse Exponential (IE)	1	ν	1	1

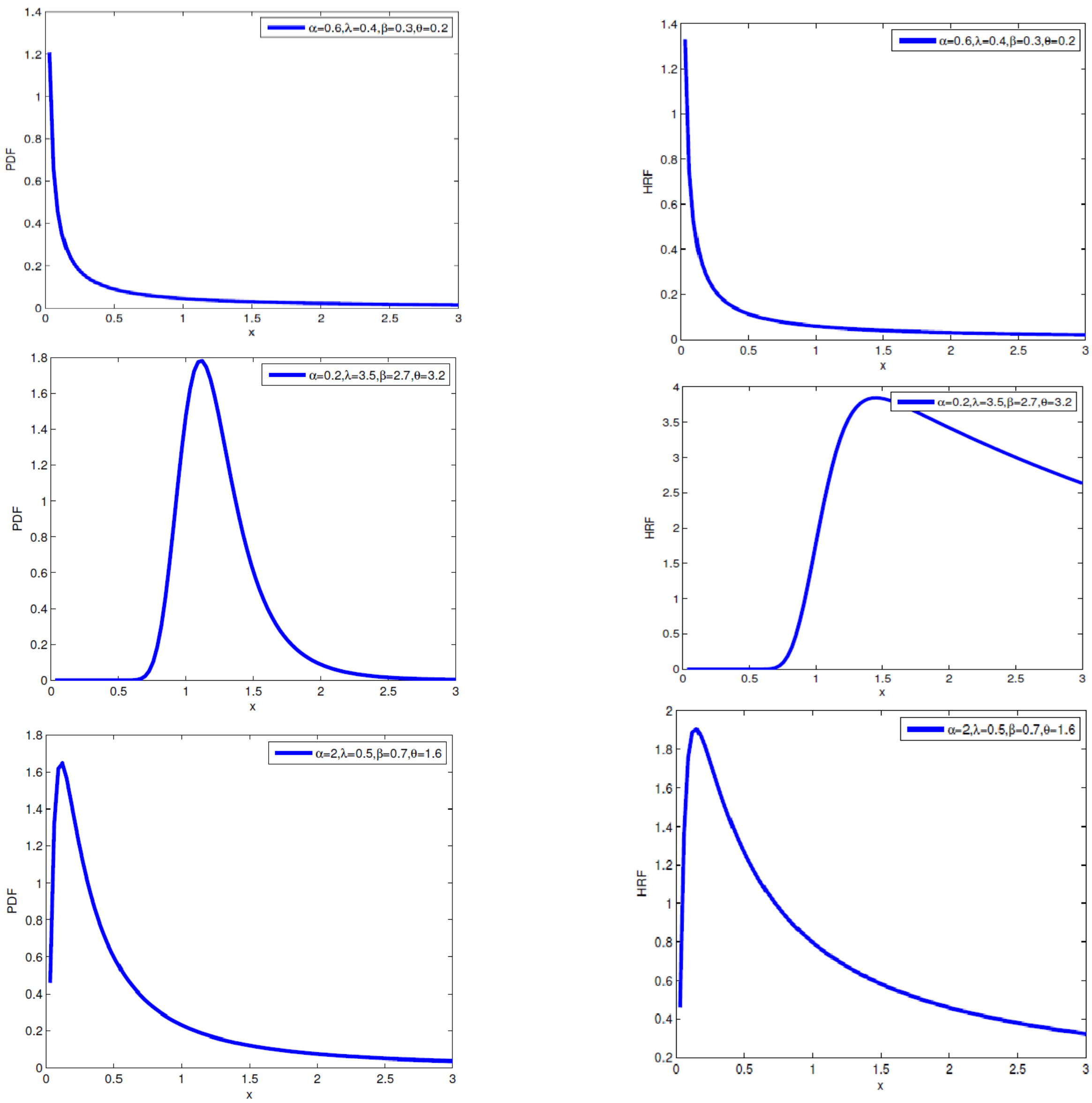


Fig. 1: Plot of the PDF of the APIEW distribution for some values of parameters**Fig. 2:** Plot of the HRF of the APIEW distribution for some values of parameters

Table 2: Some reliability of APIEW for selected values of $\lambda = 1.3$ and $\beta = 5$ at $t = 0.8$

α	ξ	HRF	MRL	RHR	MIT	SMIT
0.3	1.5	1.21717	0.24858	24.2596	0.032402	0.050077
	2.3	1.85296	0.18546	23.8596	0.032640	0.050439
0.8	1.5	0.79804	0.30161	24.5992	0.032202	0.049775
	2.3	1.22173	0.22266	24.3706	0.032337	0.049977
1.4	1.5	0.60654	0.33461	24.7944	0.032088	0.049602
	2.3	0.93246	0.24550	24.6654	0.032162	0.049714
2.6	1.5	0.43518	0.37224	25.0115	0.031963	0.049412
	2.3	0.67272	0.27133	24.9943	0.031971	0.049424

Mean Residual Life

The MRL function is defined as follows:

Strong Mean Inactivity Time

The Strong Mean Inactivity Time (SMIT) represents a $\mu(t) = \frac{1}{R(t)}\int_t^{\infty} xg(x)\,dx - t, t \geq 0$ novel reliability metric introduced by the work by Kayid and Izadkhah (2014). The definition of the SMIT:

Proposition 3.1. The MRL function for a lifetime random variable X following the APIEW distribution can $M(t) = \frac{1}{G(t)}\int_0^t 2xG(x)\,dx = t^2 - \frac{1}{G(t)}\int_0^t x^2 g(x)dx, t \geq 0$

be expressed as:

The SMIT function of APIEW distribution is:

$$\mu(t) = \frac{1}{R(t)}\sum_{m=0}^{\infty} W_m v^{\frac{1}{\eta}}(m+1)^{\frac{1}{\eta}}\gamma\left(1-\frac{1}{\eta}, v(m+1)t^{-\eta}\right) - t, \eta > 1 \quad (13)$$

$$M(t) = t^2 - \frac{1}{G(t)}\sum_{m=0}^{\infty} W_m v^{\frac{2}{\eta}}(m+1)^{\frac{2}{\eta}}\Gamma\left(1-\frac{2}{\eta}, v(m+1)t^{-\eta}\right), \eta > 2 \quad (15)$$

Proof.

By employing the definition of MRL and using (9), we get:Table (2) Provides the numerical data pertaining to HRF, MRL, RHR, and MIT (SMIT) corresponding to the specific set of selected parameters $v = 1.3$, $\eta = 5$, and $t = 0.8$ for various parameter values. α and ξ. Also, from Table (2) we see the:

$$\mu(t) = \frac{1}{R(t)}\int_t^{\infty} xg(x)dx - t$$

$$= \frac{1}{R(t)}\sum_{m=0}^{\infty} w_m \int_t^{\infty} x.\, v\eta(m + 1)x^{-(\eta+1)}e^{-v(m+1)x^{-\eta}}dx - t$$

- The decrease in HRF is observed with the increase in the MRL

Put $z = v(m + 1)x^{-\eta}$ thus

- The increases in RHR are observed with the decreases in the MIT(SMIT)

Statistical Properties

$\mu(t) = \_\_\_\_ = \sum W_m v^{-}$ $-$

$(m+1)\eta \gamma |R(t)$ $, v(m+1)t$

$|-t$ In this section, the statistical properties of the APIEW$_{m=0}$ distribution are examined, focusing on the quantile function, moments, moment generating function, entropy, and order statistics.

where, $\gamma(s,t) = \int^{t} x^{s-1} e^{-x}\, dx, s > 0$

Mean Inactivity Time

The MIT function is defined as follows:

$$m(t) = t - \frac{1}{G(t)} \int_0^t x g(x)\, dx, t \geq 0$$

Proposition 3.2. The MIT function of a lifetime random variable X with APIEW is given by:

$m(t) = t - \frac{1}{\cdot} \sum_{\eta=0}^{\infty} W\, v^{\eta} (m+1)\ \Gamma\left(1 - \frac{1}{\cdot}, v(m+1)t^{-\eta}\right), \eta > 1$ (14)Quantile Function

The solution for the quantile function of a distribution is obtained by solving the equation:

$G(x_p) = p, 0 < p < 1$ (16)

The quantile function of the APIEW distribution can be expressed as follows proposition.

Proposition 4.1. If a random variable X follows an APIEW (α, v, η, ξ) distribution, then the quantile function) $\sum m$

of X can be determined by:

Proof.

By employing the proof of the MRL and the relation

$$x_p = G^{-1}(p) = \left[\frac{-1}{v}\log\left(1 - \left(1 - \frac{\log(p\alpha - p + 1)}{\log(\alpha)}\right)^{\frac{1}{\xi}}\right)\right]^{-1/\eta} \quad (17)$$

$\Gamma(c,t) = \int_t^{\infty} x^{c-1} e^{-x} dx, c > 0$, we get the above result.

Proof.

By considering the function $h - 1 - e^{-vx^{-\eta}}$, the CDF of $Mean = \mu_1' = \mu$,

$uariance = \mu_2' - \mu^2$,

the APIEW distributionth

$G(x) = \frac{\alpha^{1-h^{\xi}} - 1}{\alpha - 1}$.

$$skewness = \frac{\mu_3' - 3\mu_2'\mu + 2\mu^3}{\left(\mu_2' - \mu^2\right)^{\frac{3}{2}}}$$ and

The p quantile function is derived by solving $G(x) = p$ and the obtained result is $h - 1 - e^{-vx^{-\eta}}$ by solving for x we get: $kurtosis =$

$$\frac{\mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4}{\left(\mu_2' - \mu^2\right)^2}$$

$$x = G^{-1}(p) = \left[\frac{-1}{v}\log\left(1 - \left(1 - \frac{\log(p\alpha - p + 1)}{\log(\alpha)}\right)^{\frac{1}{\xi}}\right)\right]^{-1/\eta}$$

Table (3) Gives the median, skewness, kurtosis and moments of APIEW distribution for specific parameters v

Statistical measures for the APIEW distribution can be calculated based on Eq. (17), such as obtaining the 1st quartile for $p = 0.25$, the median for $p = 0.5$, and the 3rd quartile for $p = 0.75$. In order to generate samples for the APIEW distribution, Eq. (17) can be utilized.

Moments

The r^{th} moments of the APIEW distribution are given= 1.3 and $\eta = 5$. along with various values of the parameters α and ξ.

Moment Generating Function

The next proposition provides the moment-generating function (MGF) of the APIEW distribution.

Proposition 4.3. If a random variable X follows an APIEW (α, v, η, ξ) distribution, then the MGF of X can be determined by:

by the following proposition. $\infty \quad \infty \quad t^r \qquad \frac{r}{\eta} \left(\quad \frac{r}{r} \right)$

$$M_X(t) = \sum\sum W_m v^{\frac{r}{\eta}} (m+1)^{\frac{r}{\eta}} \Gamma\left(1 - \frac{r}{\eta}\right), \eta > r \quad (19)$$

Proposition 4.2. If a random variable X follows an APIEW (α, v, η, ξ) distribution, then the r^{th} moments of X can be determined by:

$r=0 \; m=0 \; r!$

Proof.

We can express: $\left(\quad \frac{r}{\eta} \right)$

$\infty \qquad r \qquad r \left(\quad r \right) \qquad \infty$

$$\mu'_r = E\left(X^r\right) = \sum W \, v^{\frac{r}{\eta}} (m+1)^{\frac{r}{\eta}} \Gamma\left(1 - \frac{r}{\eta}\right), \eta > r \quad (18)$$

$$M_X(t) = \int e^{tx} g(x) dx$$

Proof. $m=0$ $\left(\eta \right)$ 0 Upon utilizing the Taylor's series expansion of the function e^{tx}, the expression simplifies to: From the definition of moments and utilizing Eq. (7), it can be derived: ∞

∞

$$M_X(t) = \sum \frac{t^r}{r!} \int_0^\infty x^r g(x) dx$$

$$\mu'_r = E\left(X^r\right) = \sum W_m v \eta (m+1) \int$$

$m=0$

Put $Z = v(m+1)x^{-\eta}$ Thus

$\infty \; x^{r-(\eta+1)} e^{-v(m+1)x^{-\eta}} dx \; 0 \; r=0$

By applying the same method used for proving moments, the result presented above is obtained.

Rényi Entropy

$\infty \quad r \quad r \left(\quad r \right)$ Rényi entropy of order δ is defined as:

$$\mu'_r = E\left(X^r\right) = \sum W_m v^{\frac{r}{\eta}} (m+1)^{\frac{r}{\eta}} \Gamma\left(1 - \frac{r}{\eta}\right), \eta > r \quad m=0 \left(\eta \right) \quad H_\delta = \frac{1}{1-\delta} \log\left(\int_{-\infty}^{\infty} (g(x))^\delta dx\right), \delta \geq 0, \delta \neq 1$$

where, $\Gamma(.)$ denotes the gamma function. Specifically, the initial two moments can be calculated as: $1 - \delta \quad -\infty$

Let $X \sim APIEW(x; \alpha, v, \eta, \xi)$ then: ∞

$$\mu'_1 = E(X) = W \, v^{\frac{1}{\eta}} \, m+1 \; \eta \; \Gamma\left(1 - \frac{1}{\eta}\right) \quad (20) \qquad H_\delta = \frac{1}{1-\delta} \log\left[\int_0^\infty \left(\frac{\log(\alpha)}{\eta} v \eta \xi x^{-(\eta+1)} e^{-vx^{-\eta}} \times \left(1 - e^{-vx^{-\eta}}\right)^{\xi - 1} \alpha^{\left(1 - e^{-vx^{-\eta}}\right)^{\xi}}\right)^\delta dx\right] \quad \sum_m$$

$m=0$ η

$= \frac{1}{} \log \sum W$

$\delta + m1) \; \delta\eta + \delta - 1\eta$

Γ And: $(\delta\eta + \delta - 1)$ $1 - \delta$ η

∞ 2 2 (2)

$=0$

$m1=0$ $\mu_2' = E\left(X^2\right) = \sum W_m v^{\eta} (m+1)\eta\, \Gamma\left(1 - \frac{2}{\eta}\right), \eta > 2$ where:

∞ $k1$

$(k1)(j1\xi + \delta(\xi - 1))(\log(\alpha))^{\delta + k1}$

$\delta - 1$

The subsequent formulas can also be utilized to $W = \sum \sum (-1)^{j1+m1}$ $v^{\eta}\, \delta^{k1} \eta^{\delta-1} \xi^{\delta}$ calculate the mean, variance, skewness, and kurtosis:

$k1=0 \; j1=0$

$(j1)(m1)$ $(\alpha - 1)^{\delta}\, k1!$

Table 3: Median and moments of APIEW for selected values of v = 1.3 and n = 5

α	ξ	Median	Mean	Variance	Skewness	Kurtosis
0.3	1.5	0.99044	1.03516	0.04331	2.25544	15.1943
	2.3	0.94313	0.96974	0.02181	1.50423	7.89711
0.8	1.5	1.04191	1.09113	0.05562	2.06334	13.3022
	2.3	0.98298	1.01049	0.02657	1.34908	7.00147
1.4	1.5	1.07581	1.12584	0.06236	1.97078	12.5658
	2.3	1.00861	1.03539	0.02894	1.26953	6.65369
2.6	1.5	1.11538	1.16533	0.06911	1.89135	12.0279
	2.3	1.03788	1.06339	0.03105	1.19840	6.41016

Table (4) presents the Rényi entropy values corresponding to the chosen parameters (v = 1.3 and η = 2.1) and various values of α, ξ, and δ. Also, from Table (4) we see that the Rényi entropy increases when the α increases (ξ decreases).by:

$\ell(x_1, \ldots, x_n \mid \Theta) = \prod_{i=1}^{n} g(x_i) = \prod_{i=1}^{n} \frac{\log(\alpha)}{\alpha - 1} v\eta\xi x^{-(\eta+1)} e^{-vx_i^{-\eta}} \left(1 - e^{-vx_i^{-\eta}}\right)^{\xi - 1} \alpha^{\left(1 - \left(1 - e^{-vx_i^{-\eta}}\right)^{\xi}\right)}$

Then, the logarithm of the likelihood function is given

Order Statistics

$L(x, \ldots, x \quad \Theta) = n\log(\log(\alpha)) - n\log(\alpha - 1) + n\log(\nu\eta\xi) - (\eta + 1) \sum_{i=1}^{n} \log(x_i)$ (24)

The order statistics of a random sample $X_1, \ldots, X_n$ refer to the ordered sample values. They are typically denoted as $X_{1:n}, \ldots, X_{n:n}$. The PDF of the i^{th} order statistic $X_{i:n}$ can be expressed as:

$-\nu\sum x_i^{-\eta} + (\xi - 1)\sum \log\left(1 - e^{-\nu x_i^{-\eta}}\right) + \log(\alpha)\sum_{i=1}^{n}\left(1 - \left(e^{-\nu x_i^{-\eta}}\right)^{\xi}\right)$

Upon deriving the first partial derivatives of the log-likelihood function with respect to the parameters in Θ, we obtain a set of equations:

$$g_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} g(x)[G(x)]^{i-1}[1 - G(x)]^{n-i}, i = 1, \ldots, n \quad (21)$$

Hence the PDF of the i^{th} order statistics $X_{i:n}$ of APIEW distribution can be obtained by substituting from (7) and (8) into (21), we get the i^{th} order statistics of APIEW density function as follows:

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha\log(\alpha)} - \frac{n}{\alpha - 1} + \frac{1}{\alpha}\sum_{i=1}^{n}\left(1 - (1 - \Psi_i)^{\xi}\right) = 0 \quad (25)$$

$$\frac{\partial L}{\partial \nu} = \frac{n}{\nu} - \sum_{i=1}^{n} x_i^{-\eta} + (\xi - 1)\sum_{i=1}^{n} \frac{\Psi_i x_i^{-\eta}}{1 - \Psi_i} - \xi\log(\alpha)\sum_{i=1}^{n} x_i^{-\eta}\Psi_i(1 - \Psi_i)^{\xi - 1} = 0 \quad (26)$$

$$g_{i:n}(x) = \frac{n!\,\nu\eta\xi\log(\alpha)}{(i-1)!(n-i)!\,\alpha - 1} x^{-(\eta+1)}\xi^{-1}\Psi(1 - \Psi)\alpha^{1-(1-\Psi)^{\xi}} \left(\frac{\alpha^{1-(1-\Psi)^{\xi}} - 1}{\alpha - 1}\right)^{i-1}\left(1 - \frac{\alpha^{1-(1-\Psi)^{\xi}} - 1}{\alpha - 1}\right)^{n-i} \quad (22)$$

$$\frac{\partial L}{\partial \eta} = \frac{n}{\eta} - \sum_{i=1}^{n}\log(x_i) + \nu\sum_{i=1}^{n} x_i^{-\eta}\log(x_i) - \nu(\xi - 1)\sum_{i=1}^{n}\frac{\Psi_i x_i^{-\eta}\log(x_i)}{1 - \Psi_i} + \nu\xi\log(\alpha)\sum_{i=1}^{n}\Psi_i(1 - \Psi_i)^{\xi - 1} x_i^{-\eta}\log(x_i) = 0 \quad (27)$$

where $\Psi = e^{-\nu x^{-\eta}}$ and

The CDF of the i^{th} order statistics $X_{i:n}$ can be expressed as:

$$\frac{\partial L}{\partial \xi} = \frac{n}{\xi} - \sum_{i=1}^{n}\log(1 - \Psi_i) - \log(\alpha)\sum_{i=1}^{n}(1 - \Psi_i)^{\xi}\log(1 - \Psi_i) = 0 \quad (28)$$

where $\Psi_i = e^{-\nu x_i^{-\eta}}$.

$$G_{i:n}(x)=\sum_{s=i}^{n}\binom{n}{s}\left[G(x)\right]^{s}\left[1-G(x)\right]^{n-s} \quad (23)$$

Due to the complexity of these equations, explicit solutions are not feasible. Therefore, numerical methods become necessary for the estimation of the MLEs of the

Hence the CDF of the i^{th} order statistics $X_{i:n}$ of APIEW distribution can be obtained by substituting from (7) into (23).

Materials and Methods

Maximum Likelihood Estimation Method

This subsection discusses the Maximum Likelihood Estimation (MLE) for the parameters $\Theta = (\alpha, \nu, \eta, \xi)$ of the APIEW distribution. Consider a complete random sample of size n from the APIEW distribution, denoted as $x_1, x_2, \ldots, x_n$. The likelihood function can be expressed as:

parameters $\Theta = (\alpha, \nu, \eta, \xi)$.

Table 4: Values of Rényi entropy of APIEW distribution

δ	α	ξ	R´enyi entropy
1.2	0.3	0.2	2.92597
		1.5	0.42441
	0.8	0.2	3.56940
		1.5	0.64175
	1.4	0.2	3.96504
		1.5	0.75677
2.3	0.3	0.2	1.99853
		1.5	0.16046
	0.8	0.2	2.46871
		1.5	0.38243
	1.4	0.2	2.77052
		1.5	0.50334

Simulation

In this subsection, we examined the behavior of MLEs derived from unspecified parameters. The simulation was carried out using the Mathematica program and the following is the technique that was followed for it:

1. Two different sets of initial parameter values are considered set(A): $\alpha = 0.6$, $\nu = 0.8$, $\eta = 0.5$ and $\xi = 0.3$ and set (B): $\alpha = 0.3$, $\nu = 0.5$, $\eta = 0.7$ and $\xi = 0.6$
2. 1000 random samples from different sample sizes $n = 50, 100, 150, 200, 300$ are generated using the (17)
3. The calculated Mean Squared Error (MSE) as well as the Bias are then presented when the result has been obtained

The MSE and bias of respective estimators are given by:model for lifetime estimation, in comparison to established distributions such as Alpha Power Inverse Weibull (APIW), Inverted Exponentiated Weibull (IEW) and Inverse Weibull (IW) distributions.

The first data set pertains to the mortality trends attributed to the COVID-19 outbreak in the United Kingdom over a span of 76 days, spanning from 15th April to 30th June 2020. This dataset was initially scrutinized by Mubarak and Al-Metwally, (2021). The second dataset delineates the durations of waiting (in minutes) prior to receiving customer assistance in a financial institution. The second dataset has been initially scrutinized by Ghitany *et al*. (2008). The first data and second data are displayed in Table (6).

The MLEs of the APIEW distribution as well as several other competing distributions are showcased in

$$MES(\Theta) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\Theta}-\Theta)^2, \; Bias(\Theta)$$

Tables (7-8) for the first and second datasets, respectively. Furthermore, Tables (7) and 8 also present

where $\Theta = (\hat{\alpha}, \hat{\nu}, \hat{\eta}, \hat{\xi})$.

$= \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\Theta}-\Theta)$

various goodness-of-fit metrics such as the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Kolmogorov-Smirnov (K-S) statistic along with their corresponding p-values for both sets of data. Analysis of these tables reveals that the

Table (5) Illustrates the Mean Squared Error (MSE) and the bias value of the parameters. Moreover, it is evident from Table (5) that the MSE (Bias) diminishes with the increment in sample size.

Real Data

In this section, we conduct an analysis of empirical data to demonstrate the efficacy of the APIEW as a viableAPIEW distribution outperformed all other competitive distributions, establishing itself as the most suitable model for fitting the provided datasets. Figures (3-4) exhibit the fitted PDFs, CDFs, RFs, and PP plots for the APIEW distribution with respect to the first and second datasets. These visual representations illustrate the capacity of the APIEW distribution to closely approximate the given datasets.

Table 5: MLE of parameters α, λ, β and θ

APIEW (α, λ, β, θ)	n	MSE($\hat{\alpha}$) Bias($\hat{\alpha}$)	MSE($\hat{\nu}$) Bias($\hat{\nu}$)	MSE($\hat{\eta}$) Bias($\hat{\eta}$)	MSE($\hat{\xi}$) Bias($\hat{\xi}$)
	50	0.637984	0.641329	0.100141	0.560831
		0.384949	0.101142	0.113329	0.118349
	100	0.586985	0.306908	0.058644	0.233698
		0.295446	-0.095049	0.093508	0.027588
APIEW (0.6,0	150	0.506365	0.235828	0.037153	0.132643
8,0.5,0.3)		0.258647-	0.079209	0.072439	-0.018376
	150	0.506365	0.235828	0.037153	0.132643
		0.258647-	0.079209	0.072439	-0.018376
	200	0.487527	0.155893	0.035711	0.065784
		0.236805	-0.050396	0.068773	-0.006879
	300	0.389554	0.113611	0.021887	0.019242
		0.219046	-0.031330	0.045049	-0.003664
	50	0.960334	1.276341	0.209416	0.704751
		0.534367	0.232847	0.172251	0.091041
	100	0.810275	0.890308	0.129319	0.672735
		0.497137	0.112601	0.141588	0.075258
APIEW (0.3,0.5,	150	0.703661	0.660812	0.102225	0.652296
0.7,0.6)		0.438629	0.085103	0.109805	0.058749
	200	0.650309	0.534065	0.078401	0.589692
		0.395532	0.061709	0.091561	0.040322
	300	0.203455	0.117308	0.026731	0.005775
		0.075293	-0.058479	0.058635	-0.009791

Table 6: The first data and second data

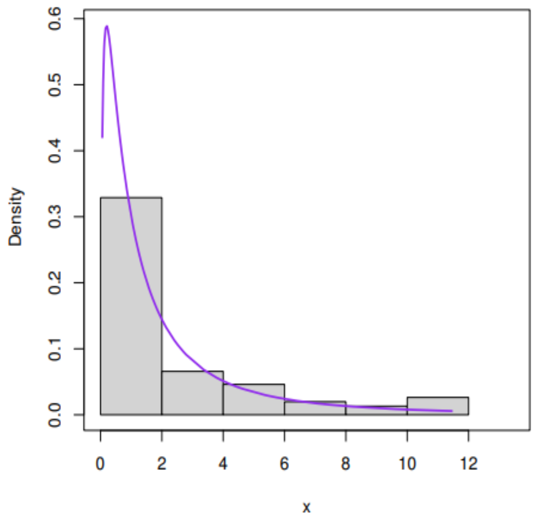
First	0.0587	0.0863	0.1165	0.1247	0.1277	0.1303	0.1652	0.2079	0.2395
data	0.2751	0.2845	0.2992	0.3188	0.3317	0.3446	0.3553	0.3622	0.3926
	0.3926	0.4110	0.4633	0.4690	0.4954	0.5139	0.5696	0.5837	0.6197
	0.6365	0.7096	0.7193	0.7444	0.8590	1.0438	1.0602	1.1305	1.1468
	1.1533	1.2260	1.2707	1.3423	1.4149	1.5709	1.6017	1.6083	1.6324
	1.6998	1.8164	1.8392	1.8721	1.9844	2.1360	2.3987	2.4153	2.5220
	2.7087	2.7946	3.3609	3.3715	3.7840	3.9042	4.1969	4.3451	4.4627
	4.6477	5.3664	5.4500	5.7522	6.4241	7.0657	7.4456	8.2307	9.6315
	10.187	11.1429	11.2019	11.4584					
Second	0.80	0.80	1.30	1.50	1.80	1.90	1.90	2.10	2.60
data	2.70	2.90	3.10	3.20	3.30	3.50	3.60	4.00	4.10
	4.20	4.20	4.30	4.30	4.40	4.40	4.60	4.70	4.70
	4.80	4.90	4.90	5.00	5.30	5.50	5.70	6.10	6.20
	6.20	6.20	6.30	6.70	6.70	6.90	7.10	7.10	7.10
	7.10	7.40	7.60	7.70	8.00	8.20	8.60	8.60	8.60
	8.80	8.80	8.90	8.90	9.50	9.60	9.70	9.80	10.70
	10.90	11.00	11.00	11.10	11.20	11.20	11.50	11.90	12.40
	12.50	12.90	13.00	13.10	13.30	13.60	13.70	13.90	14.10
	15.40	15.40	17.30	17.30	18.10	18.20	18.40	18.90	19.00
	19.90	20.60	21.30	21.40	21.90	23.00	27.00	31.60	33.10
	38.50								

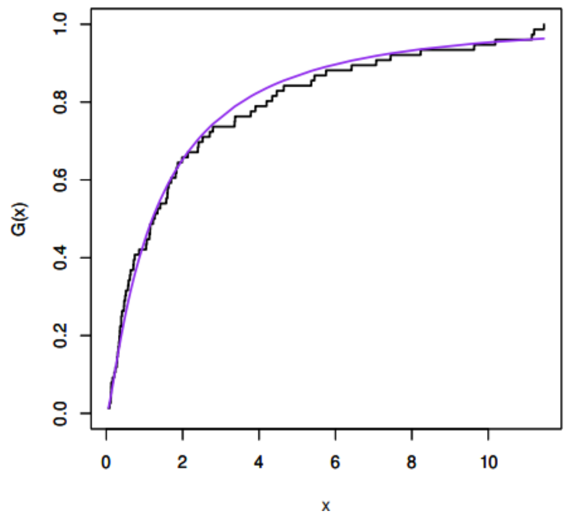
Table 7: MLEs and different statistics of APIEW for first data

	Estimates				Statistics			
Distributions	α	v	η	ξ	AIC	BIC	K-S	P-Value
APIEW	0.9186	3.805	0.242	25.18	287.538	296.861	0.0563	0.9693
APIW	17.523	0.2617	0.9450	—	292.04	299.03	0.0796	0.7208
IEW	—	1.621	0.46821	3.231	289.19	296.18	0.1009	0.4210
IW	—	0.6701	0.7896	—	294.34	299	0.1021	0.4059

Table 8: MLEs and different statistics of APIEW for second data

	Estimates				Statistics			
Distributions	α	v	η	ξ	AIC	BIC	K-S	P-Value
APIEW	32.317	4.474	1.020	2.1	652.48	662.90	0.0549	0.9231
APIW	98.324	3.076	1.487	—	658.61	666.43	0.0876	0.4261
IEW	—	7.832	1.1	1.49	668.52	676.34	0.1048	0.2218
IW	—	6.533	1.163	—	672.76	677.97	0.1166	0.1314





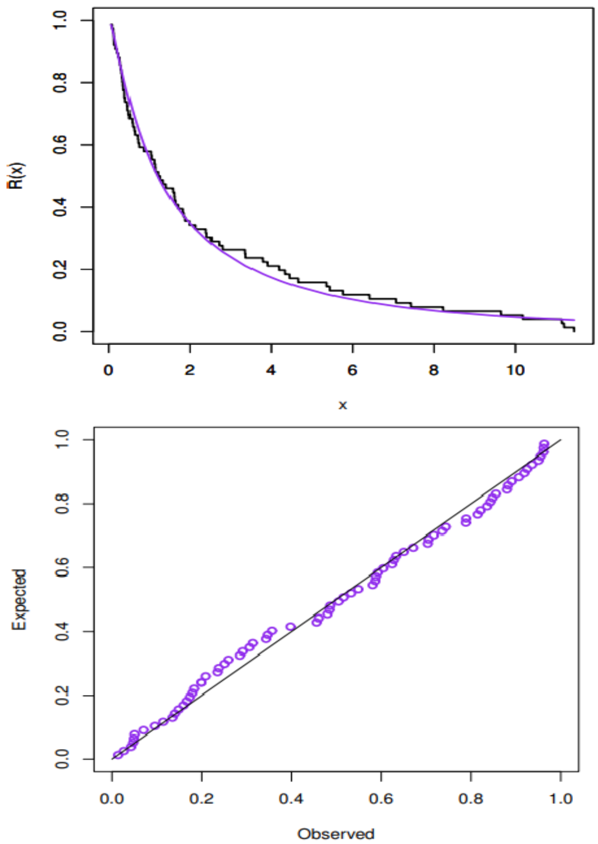
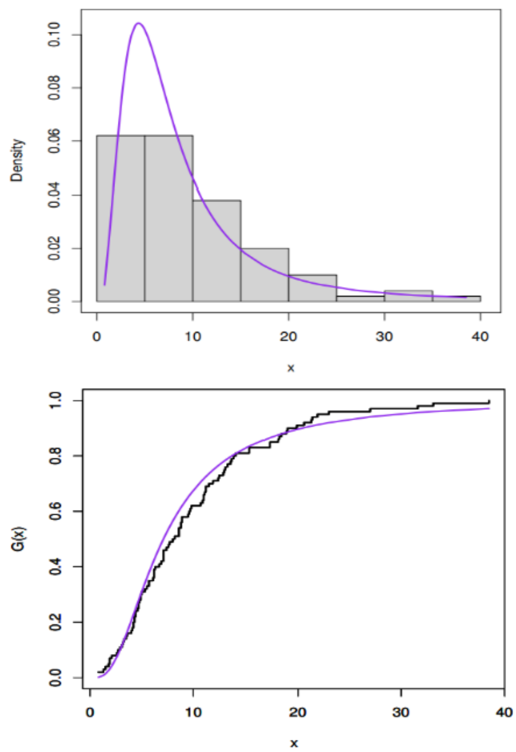


Fig. 3: Plots of the fitted functions for the APIEW distribution and PP plot for the first data



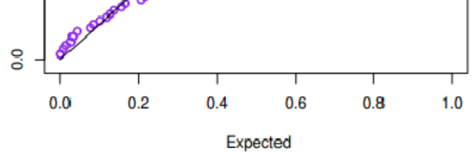


Fig. 4: Plots of the fitted functions for the APIEW distribution and PP plot for the second data

Results and Discussion

Using the maximum likelihood estimation approach, our model included four parameters, as shown in Table (5). The results demonstrated a minimal mean squared error (MSE) and bias for these parameters. Tables (7-8) and Figs (3-4) demonstrate that our suggested model is well-suited to the actual data that we used. So, it seems like our model works better with this dataset than a lot of other models out there.

Conclusion

Recent studies have made a big splash by presenting a new extended distribution that takes the Inverted Exponentiated Weibull (IEW) distribution and applies the Alpha Power (AP) transformation to it. The APIEW distribution is an expansion of the IEW distribution that has just been developed. A number of APIEW distribution statistical features have been collected and discussed.

and so on, including hazard rate, mean residual, quantile, moments, Rényi entropy, order statistics, and Rényi entropy. In addition, a simulation research confirmed that the maximum likelihood estimation approach is successful for parameter estimation, therefore it has been proposed to use it to estimate the APIEW distribution's parameters. The model's practical applicability has been shown by its effectiveness in two actual datasets. Out of all the models and distributions that have been established lately, the one that has been suggested is the one that works best for fitting these types of datasets. Research in the future may look at the difficulties with estimating using the suggested model with progressive type II censoring. The most common approaches to parameter estimation, including maximum product of spacing and least squares, may also be compared using squared error loss and LINEX loss functions.

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