MJ MULTISCIA
JOURNALS PUBLISHERS

FRONTIERS IN MATHEMATICS AND BIOSTATISTICS

ISSN: (3065- 4297)

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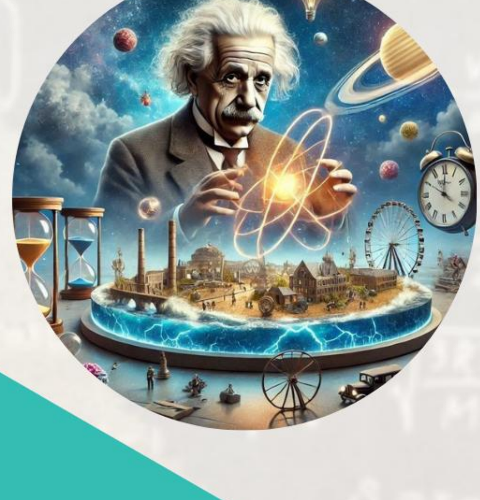

Frontiers in Mathematics and Biostatistics
Volume1, Issue3, 2025

Consideration of Boundary Value Constraints in P.D.E.s and Their Variational Symmetries

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Article Info

Received: 30-07-2025 Revised:08-08-2025 Accepted:18-08-2025 Published:28-08-2025

Abstract: In this study, an important link between Optimization theory, Trace theory, and Lie Symmetry theory is highlighted, as an extension of well-known conclusions from Partial Differential Equation (P.D.E) analysis. One important step in developing the ideas and conclusions is to re-evaluate the accepted variational symmetries with compatible Boundary Value restrictions. The methods of classical variational formulation are examined for a selection of boundary value problems. Additional important features that have been addressed to a satisfactory degree include the consequences of standard and modified admitted variational symmetries on the simplification of PDEs.

Keywords: Boundary Value Problems, Calculus of Variations, Classical Lagrangians, Symmetry Invariant Solutions, Laplace's Equation, Poisson Equation

Introduction

The usefulness of allowed variational symmetries in differentiating differential equations should be immediately considered: they play a key role in lowering the order of these equations by a factor of two (Springer Nature, 1990). When one considers that many P.D.E.s with real-world and industrial applications are derived from the calculus of variations, the significance of factors arising from this perspective becomes much more clear and persuasive. The coordinates on Ω are denoted by $x = (xi)n$, and $a(x)$ is a changing functional operating on Ω. In the case of a sufficiently regular functional $E = \int A(x,\ a,\ \nabla a)d\mu$, is a pseudo-Riemannian submanifold of $\mathbb{R}n$, and $y = (xi)n$ represents the coordinates on . We may express the multivariate Euler-Lagrange equations by perturbing x and a using the calculus of variations: In order to calculate the functions $\bar{a}$ that satisfy the optimality constraints of E, we need a P.D.E. or set of P.D.Es (Opara, 2020). Here are the equations: n Jost and Li-Jost (1998) provide an alternate equivalent formulation for the Euler-Lagrange equations. Further investigation into situations where Ω has a non-trivial topological border is therefore required, while optimization theorems that ensure the existence and/or uniqueness of the optimizing function $\bar{Z}$ already exist. The reason behind this is that when we begin with the reverse's Euler-Lagrange equations and use the Lax-Milgram theorem and Green's theorems to formulate the functional **f**, we usually don't get the same functional **f** in the first case. After plugging in the reverse's Euler-Lagrange equations into the Lax-Milgram formulation procedure, there is usually an additional trace boundary term that is realized. We will tackle this finding thoroughly by using some standard techniques from the fields of Lie Group theory and Sobolev Space theory. Examining Variational Symmetries that Are Consistent with Boundary Value Limits in Well-Posed P.D.E. In this work, two popular and easy elliptic boundary value problems (B.V.P.) will be used, drawing on prior understanding of classical basic solutions to construct the present important findings and

$F\,(x,\bar{v},\nabla\bar{v}) = \sum$ F ∂

$(x,\bar{v},\nabla\bar{v})$

propositions. For the first B.V.P, we take Laplace's v

$i{=}1$

$\overline{\partial x_i}$

$_{v_{xi}}$equation on $\mathbb{R}^2$ with the following specified constraints:

$\frac{\partial^2 u}{\partial x^2}$ $\frac{\partial^2 u}{\partial y^2}$ $[\frac{\partial^2 u}{} + \frac{\partial^2 u}{} = 0]$, that is, (P) without boundary $P_1)_{\partial x^2} + {}_{\partial y^2} = 0$ $in\, B(0,2)$
$u(x,y)|_{\partial B(0,2)} = ln8$

$3(x,y)$

1

constraints. Computing a group invariant solution with this observation, we obtain $u(x,y) = k.ln(x^2 + y^2)$.

$\{\nabla u(x,y)|_{\partial B(0,2)} = \frac{}{4}$

The solution to (P_1) is $u(x,y) = \frac{3}{2}ln(x^2 + y^2)$, whichThe manifold chosen in this setting is the ball $B(0,r)$ due to the convenience of evaluating the group invariant solution, as confined to its boundary. For $k = \frac{3}{2}$ and $r = \frac{}{2}$
is compatible with the infinitesimal symmetry $y\frac{\partial}{\partial x} -$
$x\frac{\partial}{\partial y}$: a generator of the rotation group on the Riemannian
manifold $B(0,2) \subset \mathbb{R}^2$. This infinitesimal generator is also a variational symmetry of Laplace's equation in the usual sense, without the imposition of boundary constraints.

For the second B.V.P, we take Poisson's equation, as given below:

$$(P_2)\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f \ \ in\, \Omega \\ u(x,y)|_{\partial\Omega} = 0 \end{cases}$$

The set Ω above is a bounded open subset of $\mathbb{R}^2$ with a C^1 topological boundary, and f is a fixed function in the Hilbert Space $L^2(\Omega)$. The solution to (P_2) may be compatible with some infinitesimal symmetry of Laplace's equation: $\xi\frac{\partial}{} + \frac{\partial}{}$
$+ \eta\frac{\partial}{}$, given that f satisfies
2, the B.V.P (P_1) is realized (although generalization to
arbitrary positive constants k and r would equally corroborate findings). We shall show computationally that the same rotational symmetry known to be admitted without boundary constraints is admitted as a variational symmetry by the B.V.P (P_1) , by implementing the standard vector prolongation technique and Stokes' Theorem. Before commencing computations, we reckon that the weak formulation of this equation is carried out in the Hilbert space $H^1(B(0,2))$. We need to make reference to the classical Lax-Milgram theorem stated below to formulate the optimized functional required, as the integral of a Lagrangian.

Lax-Milgram theorem: (Brezis, 2011.) Assume that
$a(u,v)$ is a continuous and coercive bilinear form on a Hilbert Space H. Then given any $\varphi \in H^*$, there exists a unique $u \in H$ such that:

$$a(u,v) = \langle\varphi, v\rangle \ \forall v \in H.$$

∂y ∂u

the first-order linear P.D.E :Moreover, if a is symmetric, then u is characterized by:$\partial f \partial f \partial\eta \partial\xi \partial\tau$

$$\xi\frac{}{\partial x} + \tau\frac{}{\partial y} = (\frac{}{\partial u} - \frac{}{\partial x} - \frac{}{\partial y}).f \qquad (P_3)\frac{1}{2}a(u,u) - \langle\varphi, u\rangle = \min_{v\in H}\{\frac{1}{2}a(v,v) - \langle\varphi, v\rangle\}$$

The said infinitesimal generator is a (pseudo-)
variational symmetry of Poisson's equation (P_2) given that η is a constant multiplied by u.

The P.D.E's (without boundary constraints) in (P_1)

Now, $\forall v \in H^1(B(0,2))$, we have the following:
and (P_2) are obtained as Euler-Lagrange equations in the $\partial^2 u$ $\partial^2 u \partial^2 u \partial^2 u$
process of optimizing specific `Energy Functionals'

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \implies \int_{B(0,2)} (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}).v = 0$$

expressed as integrals of Lagrangians. When these Euler- Lagrange equations are implemented in formulations from the Lax-Milgram theorem and Green's theorems,

$$\implies \int_{B(0,2)} (\frac{\partial u}{\partial x}.\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.\frac{\partial v}{\partial y})$$

then additional (trace) terms emerge to modify the original Energy functions. We shall refer to these modified functionals as the 'Total Energy Functionals'. It is a worthwhile venture to identify how the boundary trace terms in these modified functionals are extensions from interiors of the manifolds of the definition of the B.V.P's in such a way as to maintain the admittance of variational∂h

$$\{ \quad -\int_{\partial B(0,2)} . v d\sigma = 0 \qquad \overline{\partial N}$$

The final line above is determined from Green's formula for multivariate integration. The function h is the restriction of u to $\partial B(0,2)$, N is the Gauss map on $\partial B(0,2)$, and $\partial h = \langle \nabla h, N \rangle$. The surface element $d\sigma$ is
symmetries by the modified functionals.

obtained as:∂N

Materials and Methods

Computational Procedures for Confirmation of Variational Symmetries

$$d\sigma = n_1 dy - n_2 dx, \quad where\ N = (n_1, n_2)$$

Applying the Lax-Milgram theorem with the symmetric, continuous, and coercive bilinear form
For Laplace's equation (P_1), the infinitesimal rotation
$a(u, v) = \int \nabla u . \nabla v$on $H^1(B(0,2))$; and the $\partial B(0,2)$
roup generator $y \overline{\quad} - x \overline{\quad}$ is admitted by the equation∂x ∂yfunctional φ in the dual of $H^1(B(0,2))$ given by:

$$\varphi(v) = \int \frac{\partial h}{} . v d\sigma , \partial \quad \partial \quad \partial$$

$$(\xi \quad + \tau \quad + \eta$$

$$+ \eta^x \frac{\partial}{\partial u_x} + \eta^y \frac{\partial}{\partial u_y})[L]_{\partial B(0,2)} \partial N \partial x \partial y$$

∂u

$$+ L.(D_x\xi + D_y\tau) = 0 \iff$$

we establish the unique existence of a solution u to
$\xi(-\frac{3}{2}u.x - \frac{3}{16}u_x(x^2+y^2) - \frac{3}{8}x(x.u_x + y.u_y))$
(P_1) in $H^1(B(0,2))$. We also importantly establish that:

$$+ \tau(-\frac{3}{2}u.y - \frac{3}{16}u_y(x^2+y^2)$$

$$u = \min\{\frac{1}{2}\int_{B(0,2)} \|v\|^2 - \int_{\partial B(0,2)} \frac{\partial h}{\partial N} . v d\sigma\}_{\in H} . - \frac{3}{8}y(x.u_x + y.u_y)) + \eta(-\frac{3}{4}(x^2+y^2))$$

We now have to express the total energy functional within curly brackets in the above optimization problem under one common integral over the Riemannian manifold $B(0, 2)$ with the aid of Stokes' theorem. For (P_1), $d\ \sigma$ is easily computed, as the Gauss map (outward unit normal) here is simply $N = (\frac{x}{2}, \frac{y}{2})$. Hence, the

$$+ (D_x\eta - u_x D_x\xi - u_y D_x\tau)(u_x - \frac{3}{16}x(x^2+y^2))$$

$$+ (D_y\eta - u_x D_y\xi - u_y D_y\tau)(u_y - \frac{3}{16}y(x^2+y^2))$$

differential one-form $\omega = \frac{\partial h}{\partial N} . v d\sigma$ is computed:$-\frac{3}{16}[4u(x^2+y^2) + (\frac{1}{2}u_x^2 + \frac{1}{2}u_y^2)$

$$\omega = \langle \nabla h, N \rangle \, v d\sigma = \frac{3}{8}(x^2 + y^2).v.\left(-\frac{x}{2}dy - -\frac{y}{2}dx\right).$$

$+ (x^2 + y^2)(xu_x + yu_y)])(D_x\xi + D_y\tau) = 0$

Its exterior derivative $d\omega$ is thereby computed as:For the infinitesimal symmetry criterion to be met as expanded above, we must have the coefficients of $\{u_x, u_y,$ their powers, and products of$\}$ to be zero. Moreover, the sum of all terms free of $u_x$and u_y must also equal zero. We thereby determine the following table to evaluate the coefficients of these identified monomials. (Note that $D_x\xi = \xi_x + \xi_u u_x$, $D_y\xi = \xi_y + \xi_u u_y$ and so on.)

$$d\omega = \frac{3}{16}[4v(x^2 + y^2)dx \wedge dy + (x^2 + y^2)(xv_x + yv_y)dx \wedge dy]$$

From Stokes' theorem, we have $\int_{\partial B(0,2)} \omega = \int_{B(0,2)} d\omega$, giving us a total energy functional here to be:

$$\int_{B(0,2)} \left(\frac{2}{2}\|v\|^2 - \frac{3}{16}[4v(x^2 + y^2) + (x^2 + y^2)(xv_x + yv_y)]\right) dx \wedge dy \longrightarrow (E_1).$$

Monomials	Coefficients
Free	$-\frac{3}{2}ux\xi - \frac{3}{2}uy\tau - \frac{3}{4}x^2\eta - \frac{3}{4}y^2\eta - \frac{3}{16}x^3\eta_x - \frac{3}{16}xy^2\eta_x - \frac{3}{16}yx^2\eta_y - \frac{3}{16}y^3\eta_y - \frac{3}{4}ux^2\xi_x - \frac{3}{4}uy^2\xi_x - \frac{3}{4}ux^2\tau_y - \frac{3}{4}uy^2\tau_y$
${u_x}^2$	$\eta_u - \frac{1}{2}\xi_x + \frac{1}{2}\tau_y$
${u_x}^3$	$-\frac{1}{2}\xi_u$
$u_x.u_y$	$-\tau_x - \xi_y$
$u_x.{u_y}^2$	$-\frac{1}{2}\xi_u$
${u_x}^2.u_y$	$-\frac{1}{2}\tau_u$
${u_y}^2$	$\eta_u + \frac{1}{2}\xi_x - \frac{1}{2}\tau_y$
${u_y}^3$	$-\frac{1}{2}\tau_u$

Assume that $(E_1$ admits a variational symmetry $\mathbf{v} = [\xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u}]$.

The infinitesimal variational symmetry criterion is hereby given as:

$$pr^{(1)}\mathbf{v}[L] + L.div(\psi) = 0 \iff \left(\xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} + \eta^x\frac{\partial}{\partial u_x} + \eta^y\frac{\partial}{\partial u_y}\right)[L] + L.(D_x\xi + D_y\tau) = 0,$$

where L is the Lagrangian of (E_1) with v replaced by u, $\psi = (\xi, \tau)$ and, $\eta^x = D_x\eta - u_xD_x\xi - u_yD_x\tau$, $\eta^y = D_y\eta - u_xD_y\xi - u_yD_y\tau$. For details on how the infinitesimal symmetry criterion is obtained, we refer the reader to (Springer Nature, 1990). Developing the formulation of the infinitesimal symmetry criterion for this case, we have:

u_x $\quad -\frac{9}{16}x^2\xi - \frac{3}{16}y^2\xi - \frac{3}{8}xy\tau + \eta_x$
$- \frac{3}{16}\eta_u(x^3 + xy^2)$
$+ \frac{3}{16}\xi_y(y^3 + yx^2)$
$- \frac{3}{16}\tau_y(x^3 + xy^2)$

u_y $\quad -\frac{9}{16}y^2\tau - \frac{3}{16}x^2\tau - \frac{3}{8}xy\xi + \eta_y$
$- \frac{3}{16}\eta_u(y^3 + yx^2)$
$+ \frac{3}{16}\tau_x(x^3 + xy^2)$
$- \frac{3}{16}\xi_x(y^3 + yx^2)$

not alter the original Euler-Lagrange equation, in view of the strict variational symmetry present in this instance. A more elaborate description of the concept of alteration of Euler-Lagrange equations obtained from total energy functionals in cases of strict and pseudo-variational symmetries would enhance this practical theory quite richly. In what ensues, analysis of the second referenced B.V.P (P_2) introduces a proposed prospect for further development of this concept.

For Poisson's equation (P_2), let $\psi(x, y)$ be a harmonic function, that is, a solution to (P_1) without its boundary constraints. The generic infinitesimal group generator:

$$\mathrm{v} = k\ [\alpha\ (x,y)\frac{\partial}{\partial x} + \beta\ (x,y)\frac{\partial}{\partial y}] + (ku + \psi(x,y))\frac{\partial}{\partial u}$$

Results

Upon equating the coefficients from the above table to zero, we can conclude with relative ease that the infinitesimal variational symmetry criterion for (P_2) is

$$:= \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u}$$

is admitted by the equation $[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f]$ in (P_2),

met if and only if ξ = y, τ = -x, η = 0. This is precisely where $\{k_j\}$'s and k are arbitrary real constants and the rotational symmetry that was identified as being compatible with the solution to (P_1). This is an exemplary computation with a result that may be suitably generalized to a similar B.V.P of Laplace's equation, as pointed out earlier. As a noteworthy remark, (P_1) without boundary $\{\alpha_j(x,y) + i.\beta_j(x,y)\}$ constitute the collection of analytic complex-valued functions ($i = \sqrt{-1}$), given that f satisfies the first order linear P.D.E (P_3):

$$\xi\frac{\partial f}{\partial x} + \tau\frac{\partial f}{\partial y} = (\frac{\partial \eta}{\partial u} - \frac{\partial \xi}{\partial x} - \frac{\partial \tau}{\partial y}).f$$

value constraints is famously the Euler-Lagrange equation for optimizing the Dirichlet Energy functional, that is:

$$\frac{1}{2}\int_\Omega \|\nabla v\|^2 = \frac{1}{2}\int_\Omega (v_x^2 + v_y^2)\ : v \in H^1(\Omega)$$

On the other hand, beginning with the Euler-Lagrange equation to formulate the functional which it optimizes brings up an additional boundary term via Green's theorem (analogous to a constant of integration in single variable integration) which can be processed further via Stokes' theorem for the purpose of performing the

This observation is made by direct application of the infinitesimal symmetry criterion (Springer Nature, 1990):

$$pr^{(1)}\mathrm{v}[u_{xx} + u_{yy} - f] = 0 \quad \text{whenever} \quad u_{xx} + u_{yy} - f = 0$$

for any admissible infinitesimal generator $\mathrm{v} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u}$. The Lie Algebras spanned by $\psi(x,y)\frac{\partial}{\partial u}$ and $[\alpha_j\ (x,y)\frac{\partial}{\partial x} + \beta_j\ (x,y)\frac{\partial}{\partial y}]$ are identified as infiniterequisite vector field prolongation. It is this modification of the Dirichlet Energy we hereby refer to as the Total Energy Functional. One may desire to view the outcome of obtaining the Euler-Lagrange equations from this functional in (E_1), as done below:dimensional sub-algebras of the overall admitted infinitesimal symmetry.

Poisson's equation is determined as the Euler- Lagrange equation for optimizing the functional:

$$\int_\Omega (\frac{1}{2}\|v\|^2 - v.f)\,dx \wedge dy := \int_\Omega L\,dxdy \longrightarrow (E_2)$$

$$L_v(x,\bar{v},\nabla\bar{v}) = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} L_{v_{xi}}(x,\bar{v},\nabla\bar{v})$$

$$\Rightarrow -\frac{3}{4}(x^2+y^2) = \frac{\partial}{\partial x}[v_x - \frac{3}{16}x(x^2+y^2)] + \frac{\partial}{\partial y}[v_y - \frac{3}{16}y(x^2+y^2)]$$

If in addition $\psi(x,y) \equiv 0$ above, then the aforementioned symmetry admitted by Poisson's equation is a pseudo-variational symmetry of the Lagrangian in (E_2), in the sense that:

$$\Rightarrow -\frac{3}{4}(x^2+y^2) = v_{xx} - \frac{9x^2}{16} - \frac{3y^2}{16} + v_{yy} - \frac{9y^2}{16} - \frac{3x^2}{16}$$

$$\Rightarrow -\frac{3}{4}(x^2+y^2) = v_{xx} + v_{yy} - \frac{12}{16}(x^2+y^2)$$

$$\Rightarrow v_{xx} + v_{yy} = 0$$

$$pr^{(1)}v[L] + L.div(\xi,\tau) = 2k.L$$

Clearly, accurate incorporation of the trace boundary portion in the Total Energy Functional for this case doesfor this sub-case. In other words, those one-parameter infinitesimal generators combining just the vectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are the strict variational symmetries ofdetect that $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ is again a (strict) variational variational problem (E_2). $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are identified withsymmetry for this total energy functional. Setting $[k = ln2 - 1]$ here, the Euler-Lagrange equation from this the unit canonical basis vectors in the axial directions of x and y respectively.

For a well-posed system with (P_2), we introduce atotal energy functional yields the following: second boundary value constraint for the gradient of the dependent variable u. We hereby fix the function f here to be $f(x,y) = ln(x^2 + y^2)$, to fit in with the admissibility of symmetries identified above, while we maintain the manifold of definition to be $\Omega = B(0,2)$ as in (P_1) for convenience of computations. Hence, we set:

$$L_v(x,\bar{v},\nabla\bar{v}) = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} L_{v_{xi}}(x,\bar{v},\nabla\bar{v})$$

$$\Rightarrow -k(x^2+y^2) + ln(x^2+y^2) = \frac{\partial}{\partial x}[v_x - \frac{k}{4}(x^2+y^2).x] + \frac{\partial}{\partial y}[v_y - \frac{k}{4}(x^2+y^2).y]$$

$$\Rightarrow -k(x^2+y^2) + ln(x^2+y^2) = v_{xx} - \frac{kx^2}{2} - \frac{k}{4}(x^2+y^2) + v_{yy} - \frac{ky^2}{2} - \frac{k}{4}(x^2+y^2)$$

$$\nabla u(x,y)|_{\partial\Omega} = (ln2 - 1)(x,y)$$

and with these additional specifications, (P_2) has a solution:

$$u(x,y) = \frac{1}{4}(x^2 + y^2 - 4)(ln(x^2+y^2) - 2),$$

which is compatible with the same infinitesimal rotation

$$\Rightarrow v_{xx} + v_{yy} = ln(x^2+y^2)$$

Accurate incorporation of the trace boundary portion in the Total Energy Functional for this case does not alter the original Euler-Lagrange equation, in view of the strict variational symmetry present. The Euler-Lagrange equations for both (P_1) and (P_2) are of the self-adjoint type, which greatly simplifies the concept of determination of all associated Lagrangians hereby considered. Non-self-adjoint differential equations group generator: $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ as seen in (P_1).

introduce further intricacies in the development of $\partial x \partial y$ [1] concepts of this sort. Discussions to ensue in the

Now, beginning with the Euler-Lagrange equation for (E_2) to weakly formulate the total energy functional which it optimizes, for all $v \in H^1(B(0,2))$, we have the following:

subsequent section present a summarization [0] of motivations and technicalities stemming from the above, including computational illustrations, galvanizing their theoretical and practical utility.

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f \implies \int_{B(0,2)} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - f \right) . v = 0$

Discussion

u $\partial v \partial u$ ∂v

Classically, there are three types of P.D.E, namely:

$\implies \int_{B(0,2)} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y}) - \int_{B(0,2)} f . v + \int_{\partial B(0,2)} \frac{\partial h}{\partial N} . v d\sigma = 0$ $- \int$ (

elliptic, parabolic, and hyperbolic; of which the prototypical equations are respectively-Laplace's equation, the heat equation, and the wave equation (Braun, 1993). Despite the limitless solutions to these

In the final line above due to Green's theorem, h is the restriction of u to $\partial B(0, 2)$. Via engagement of the Lax-Milgram theorem, we deduce that u is characterized by:equations prior to the appropriate imposition of boundary value constraints, more emphasis is laid on the `natural' fundamental solutions, which are usually supple to symmetry invariance techniques. Pertaining to Laplace's

$u = \min_{v \in H_0^1(\Omega)} \{ \frac{1}{2} \int_{\Omega} \|v\|^2 - \int_{\partial \Omega} \frac{\partial h}{\partial N} . v d\sigma + \int_{\Omega} f . v \}$ equation in two independent variables, this P.D.E is of

peculiar interest because of the well-known link between its solutions and analytic complex-valued functions.

Recalling the identity: $\frac{\partial h}{\partial N} = \langle \nabla h, N \rangle$ with the Gauss

Moreover, as pointed out in an included computational

map $N = (\frac{x}{2}, \frac{y}{2})$ on $\partial \Omega$ and implementing Stokes' result above, there is a bijective correspondence between admissible symmetries that are in terms of the canonical

theorem as in (P_1), we arrive at the total energy functional

for this case:basis vectors in the axial directions of the two independent variables of Laplace's equation, and the collection of analytic complex-valued functions. Another noteworthy

$\int_{B(0,2)} \left(\frac{1}{2} \|v\|^2 - \frac{1}{4} (ln2 - 1)(x^2 + y^2)(4v + xv_x + yv_y) + v . ln(x^2 + y^2) \right) dx \wedge dy$ property of Laplace's equation is its admittance of two

separate infinite-dimensional Lie sub-algebras as symmetries, with the sub-algebra addressed just above being key in generating invariants for the equation's simplification. Among other prospective concepts, this

We may engage the infinitesimal symmetry criterion for variational symmetries as done previously for (P_1) tosuggests a viable platform for the study of the special embeddings of $C^{\infty}(\mathbb{C},\mathbb{C})$ in larger complex Sobolev

spaces; as a vantage point for analysis of analytic complex-valued functions, made accessible via standard point symmetries of Laplace's equation. We should note that these symmetries are identical to what is admissible by Poisson's equation, given fulfillment of the earlier identified compatibility first-order P.D.E (P_3); observing that Poisson's equation is the non-homogeneous counterpart of Laplace's equation.

Before equivalent ideas may emerge from parabolic P.D.E. analysis, it is essential that the aforementioned computational findings be consistent with the officially published literature on classical elliptic P.D.E. The use of $\mathbf{O}\infty$ functional formulation in symmetry approaches seems to conflict with the Hilbert spaces of formulation required by the theorems of existence and uniqueness of solution that are mentioned as fundamental, suggesting a possible change in emphasis. It is true that there is a fine line to walk when moving from Hilbert Sobolev spaces to dense $a\infty$ functional subspaces, where symmetry methods are used, and back again. For example, it's important to note that the validity of P.D.E. assertions depends solely on almost universal correspondence to analytic functional requirements. When considering this degree of freedom, the Total Energy functional that needs optimizing to produce each Boundary Value Problem could be drastically different from what we get after using Green's theorem of integration, but only by a Radon measure. This specific discrepancy in the energy functionals is found to be an insignificant affine separation, and the more general discussion of such separations is an interesting fact that may be used to broaden the range of calculations using pseudo-variational symmetries. Divergence symmetries, variational $\mathbf{O}\infty$ symmetries, and μ −symmetries are all examples of pseudo-variational symmetries. These symmetries are just as effective as strict variational symmetries in reducing differential equations to the same degree or order.The possibility of formulating a solid theory that connects pseudo-variational symmetries with affine separations between related Lagrangians is obvious. In such situation, variational symmetries from alternative (total energy) Lagrangians might be helpful in solving certain interesting B.V.Ps, which could have important pedagogical and scientific implications. This is very relevant since, in most cases, dissipative systems (often controlled by parabolic PDEs) cannot be derived from variational problems in a strict sense, as doing so would lead to the contradicting assumption that they are tied to classical conservation rules. In addition,Although not new, the inclusion of boundary contributions to the optimized Lagrangians is recognized as an underappreciated part of Noether's theorem (Halder et al., 2018). However, this overlooked aspect could be a crucial component in validating extensions to similar techniques for pseudo-variational symmetries.

Conclusion

Researching the parallels and differences between elliptic B.V.P. and parabolic weak formulation approaches is another important and relevant future endeavor. As an example, all of the heat-type equations that describe important physical events are either parabolic or weakly parabolic. Through potential pseudo-variational formulation approaches modified from the classical expositions of this article, symmetry invariant solutions often describe equilibrium states of several such physical systems. Modifying the conventional Lax-Milgram to fit the quirks of its parabolic analogue, J.L. Lion's theorem, is likewise necessary for making the change from elliptic to parabolic B.V.P.s (Brezis, 2011). An in-depth understanding of the Trace theory of multivariate integration in Sobolev spaces is essential for the advancement of all the ideas that are proposed and established below. Everyone knows that the following map has a linear and continuous Trace Operator (γ):

$$\gamma: W^{k,p}(\Omega) \longrightarrow L^p(\partial\Omega)$$

for all Sobolev spaces $W^{k,p}(\Omega)$: $k \in \mathbb{N} \cup \{0\}$, $p \geq 1$. Consider a sufficiently regular first-order Lagrangian functional on $W^{k,p}(\Omega)$ is given by:

$$E(u) = \int_{\Omega} F(x, u, \nabla u) dV$$

for some open and bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. Elements $\bar{u}$ in the kernel of the Frechet differential of E comprise a linear subspace of $W^{k,p}(\Omega)$, such that for any

$\bar{u} \in Ker[E']$, we have:

$$\langle E'(\bar{u}), v\rangle = 0 \ \ \forall v \in W^{k,p}(\Omega) \ .$$

For all $u \in W^{k,p}(\Omega)$, $E'(u)$ can be described as a bounded linear functional from the dual space $(W^{k,p}(\Omega))^*$ of $W^{k,p}(\Omega)$. Invariably, the collection of variational-symmetry invariant solutions to the Euler- Lagrange equation(s) associated with some Lagrangian comprises a specific subspace of $Ker[E']$. As a final noteworthy remark, we find that implementation of the infinitesimal vector prolongation technique on just $L^p(\partial\Omega)$ (facilitated by pull-backs into the interior of Ω), for the illustrations considered in this paper, is sufficient to solve for any admitted symmetries. This suggests how systematic engagement of Trace theory could be relevant in symmetry analysis of P.D.E's.

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With contributions from A. K. Halder, P. G. L. Leach, and A. Paliathanasis (2018). Symmetry and Noether's Theorem. Paper number: 744 in Symmetry, volume 10, issue 12. Viewed at https://doi.org/10.3390/sym10120744
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