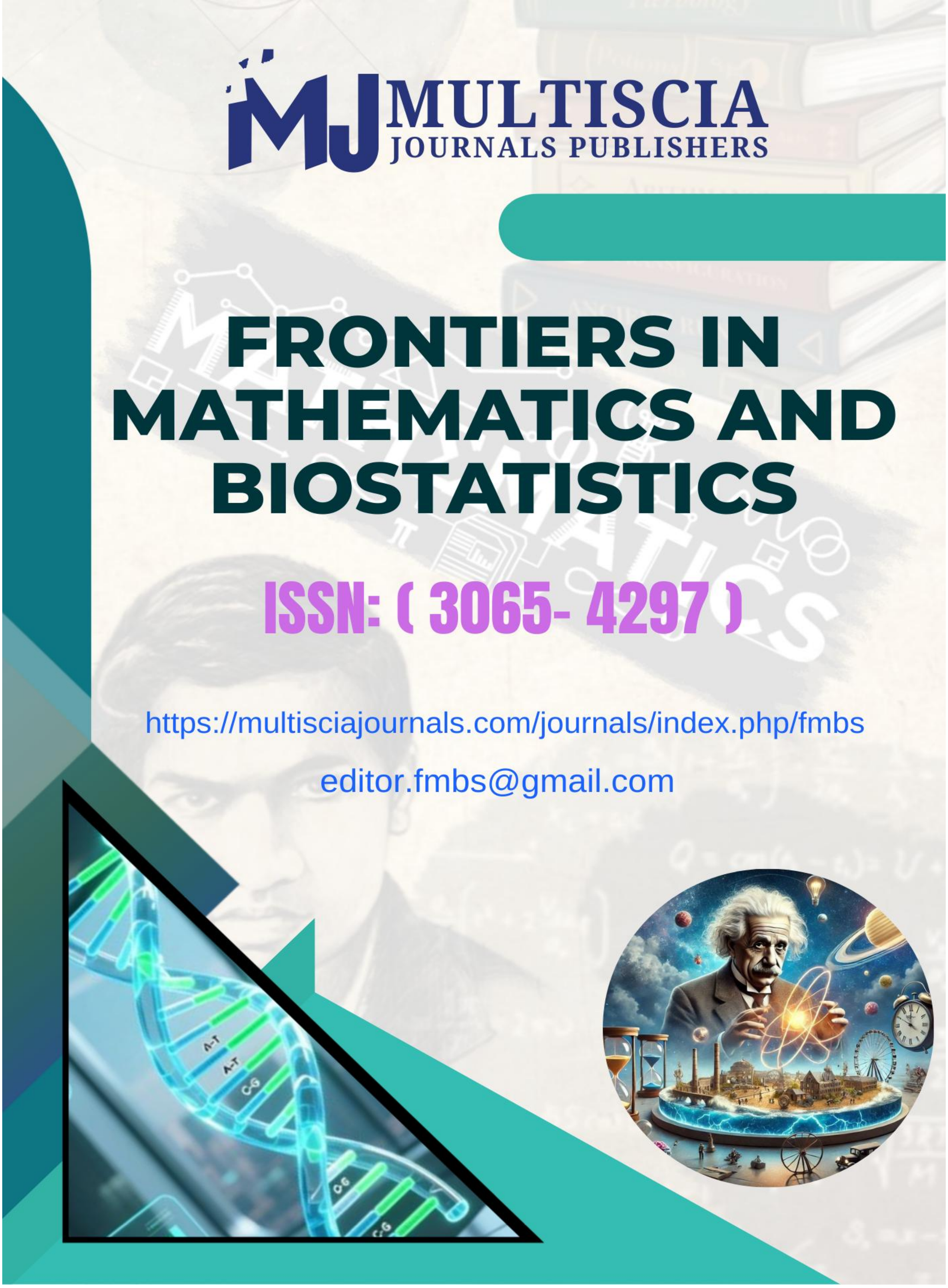

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Deformed q-statistics derived from position-dependent mass Schrödinger Eq.

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Abstract: Highway traffic congestion, characterized by its inherent instability, has been extensively studied using deterministic models, providing valuable insights. However, these models often overlook the stochastic nature of driver behavior, a key factor that significantly impacts traffic flow. Recognizing this, a car-following model with discretionary lane changes to analyze their effect on traffic dynamics was introduced. While the mathematical results were sound, the use of the Optimal Velocity Model (OVM) led to unrealistic outcomes in certain situations, such as heavy traffic jams, due to its oversimplification. To address these limitations, a car- following model incorporating human behavior through the Cox-Ingersoll- Ross (CIR) process, demonstrating that traffic instability arises from the stochastic characteristics of traffic flow was proposed. However, traffic instability can be triggered by various factors, including high lane-change rates, incivility, queue properties, and accidents. In this study, we propose an enhanced model that integrates stochastic elements into traffic flow dynamics, while retaining the key stimulus-response mechanisms. Using the Intelligent Driver Model (IDM) and incorporating the Langevin equation with stochastic behavior modeled through the Ornstein-Uhlenbeck process, we aim to provide a more realistic representation of traffic flow. The model is calibrated using the NGSIM dataset and compared with existing approaches, to evaluate its effectiveness in capturing real-world traffic phenomena. Our results highlight the significant impact of perturbations, such as moving bottlenecks, on traffic oscillations.

Keywords: Thermodynamic Properties, Deformed Exponential Function, Position-Dependent Mass

Introduction

Extensive and non-extensive properties of thermodynamic systems have been investigated in relation to entropy, or more specifically, the mathematical characteristics of exponential and logarithmic functions that directly define entropy (Nivanen et al., 2003; Wang and Le Méhauté, 2002; Sargolzaeipor et al., 2018). In statistical mechanics, an extensive function is defined as follows: if systems A and B are independent, then pij (A + B) = pi (A) pj (B), and S (A + B) = S (A) + S (B) (Wang et al., 2002). The entropy is a nonnegative, concave, and constructionally nonnegative function. The conventional logarithm and exponential functions, namely ln(AB) = ln(A) + ln(B), may be used to accomplish this mathematical feature. The expression exp(A — B) is equal to the product of exp(A) and exp(B). The non-extensive systems follow the type relationship, in contrast to the extensive ones. This type relationship allows us to identify a pseudo-additivity that would be obtained from a q-deformed exponential function engaged in what has been referred to as qalgebra with distortions (Abe, 2001). To summarize, q-algebras enable the introduction of the q-calculus, which in turn allows us to solve the original problem's describing equation in the new coordinate space or to uncover physical features that were intractable in primitive space. The generalization of the conventional exponential and logarithmic functions is where the formalism of q-deformed algebra is located. Based on this extension, a class of q-deformed algebraic operations like q-addition, q-subtraction, and q-product can be introduced (Kaniadakis, 2001; 2002; 2005) and a q-deformed differential operator (q-calculus) can be created, which can lead to a mathematical framework that is backed by a clearly defined Abelian field (Scarfone, 2015). While maintaining the key characteristics of the standard Boltzmann-Gibbs statistical mechanics, a generalization of the field based on the q-exponential function has already been shown (Tsallis, 1988). Theoretical consistency, possible applications, and foundations of q-deformed exponential functions in statistical mechanics have been the subject of many recent articles (Silva, 2006; Kim et al., 2019). There have been other potential uses as well, such as quantum

$$- \quad \frac{\hbar}{} \quad +$$

$$\left[\frac{d}{dx}\left(2m(q;x) \right)\frac{d}{dx} \right| \quad | \quad |$$

$$_{\alpha\beta}(x) \Big] \psi(x) = E\psi(x) \quad \cup$$

(3)
Tirnakli, 2020). Consequently, due to their multiple applications, the q-deformed exponential function has also been proposed in the treatment of the Position-Dependent MASS Schrödinger Equation (PDMSE), as an introduction to the concept of q-deformed quantum mechanics. Specifically, it has been related to a change in the linear momentum operator (Borges, 2004; Curado and Tsallis,where:

$$U_{\alpha\beta}(x) = V(x) + \frac{\hbar^2}{4}(\beta + 1)\frac{m''(q;x)}{m^2(q;x)}$$

$$U_{\alpha\beta}(x) = V(x) + \frac{\hbar^2}{4}(\beta + 1)\frac{m''(q;x)}{m^2(q;x)}$$

(4)
1991) which implies the existence of a relationship between the statistical mechanics and the q-deformed quantum mechanics. Indeed, it is well known that the displacement operator is directly related to the linear momentum operator through the exponential function (Costa Filho *et al.*, 2011). For that, the linear momentum operator would be generalized through the q-deformed exponential function under the formalism of the q-algebras (da Costa *et al.*, 2020). So, the q-deformed exponential function is related not only to the PDMSE for solving quantum interactions but also to the so-called q-statistics in the generalization of the additive property of the Boltzmann Gibbs (BG) entropy (Gomez and Borges, 2021). With this purpose, in the following, we begin by considering the q-deformed linear momentum operator in such a way that the canonical form of the position-dependent mass Schrödinger equation can be achieved. After that, it is presented the q-deformed quantum dynamic variables that are needed to obtain the generalization (q-deformed) of the most important thermodynamic properties. In the end, the usefulness of the proposed approach is exemplified by considering two different hyperbolic forms of position- dependent mass distributions.

The q-Deformed Quantum Mechanics

The q-Deformed Quantum Linear Momentum Operator

Starting with the position-dependent mass Schrödinger equation (Von Roos, 1983):

With the aim of transforming Eq. (3) into its canonical form, we factorize the Hamiltonian of Eq. (3) as follows:

$$\hat{H} = \frac{d}{dx}\left(\frac{i\hbar}{\sqrt{2m(q;x)}}\right)\left(\frac{i\hbar}{\sqrt{2m(q;x)}}\right)\frac{d}{dx} + U_{\alpha\beta}(x) \tag{5}$$

such that if we use the commutator:

$$\left[\frac{d}{dx}, \quad \frac{i\hbar}{\sqrt{2m(q;x)}}\right] = \left(\frac{1}{\sqrt{2m(q;x)}}\right)' \tag{6}$$

we have:

$$\hat{H} = \left[\frac{i\hbar}{\sqrt{2m(q;x)}}\frac{d}{dx} + i\hbar\left(\frac{1}{\sqrt{2m(q;x)}}\right)'\right]\left[\frac{i\hbar}{\sqrt{2m(q;x)}}\frac{d}{dx}\right] + U_{\alpha\beta}(x) \tag{7}$$

Thus, we can write:

$$\hat{H} = \left(\frac{i\hbar}{\sqrt{2m(q;x)}}\frac{d}{dx}\right)^2 - \frac{i\hbar m'(q;x)}{4m(q;x)\sqrt{2m(q;x)}}\left(\frac{i\hbar}{\sqrt{2m(q;x)}}\frac{d}{dx}\right) - \left(\frac{i\hbar}{\sqrt{2m(q;x)}}\frac{d}{dx}\right)\left(\frac{i\hbar m'(q;x)}{4m(q;x)\sqrt{2m(q;x)}}\right) - \frac{\hbar^2\left(\frac{m'(q;x)}{4m(q;x)\sqrt{m(q;x)}}\right)'}{\sqrt{2m(q;x)}} + U_{\alpha\beta}(x) \tag{8}$$

where the commutator has been used:

$$\hat{K}_{\alpha\beta}\psi + V(x)\psi(u) = E\psi \tag{1}$$

where, the operator $\hat{K}_{\alpha\beta}$ is the von Ross kinetic energy operator given by:(9)

$$\left[\frac{i\hbar}{\sqrt{2m(q;x)}}\frac{d}{dx}, \frac{i\hbar m'(q;x)}{4m(q;x)\sqrt{2m(q;x)}}\right] = \frac{i\hbar}{\sqrt{2m(q;x)}}\left(\frac{m'(q;x)}{4m(q;x)\sqrt{m(q;x)}}\right)'$$

At this point, it should be noticed that the

$$\bar{K}\alpha\beta = -\frac{h^2}{4}\left(\frac{d}{dx}m^{\beta}(q;x)\frac{d}{dx}m^{\gamma}(q;x) + m^{\gamma}(q;x)\frac{d}{dx}m^{\beta}(q;x)\frac{d}{dx}m^{\alpha}(q;x)\right) \tag{2}$$

$| m^{\alpha}(q;x)$

apostrophe refers to a standard derivative with respect to the position. Finally, we have the canonical form of the Hamiltonian:

Rego-Monteiro *et al*. (2016) given:

$${}_q = -\frac{i\hbar}{\sqrt{M(q;x)}}\frac{d}{dx} - \frac{i\hbar}{2}\left(\frac{1}{\sqrt{M(q;x)}}\right)'$$

where:

the operator $\hat{p}_q$ could not be self-adjoint if the $(\hat{p}_q, D) \neq (\hat{p}_q^{\dagger}, D')$ inequality holds. In that case, we

is the q-deformed position-dependent mass linearwould be dealing with a self-adjoint extension for the momentum operator,

$$M(q;x) = m(q;x)/m_0$$

and

$u_{eff}(x)$ is operator

$(\hat{p}_q, D)$

(Gadella *et al*., 2007). On the other

the effective potential:

$$u_{eff}(x)=\frac{\hbar^2}{2m_0}\left[\left(\left(\frac{1}{2\sqrt{M(q;x)}}\right)'\right)^2+\frac{\hbar^2}{\sqrt{M(q;x)}}\left(\frac{1}{2\sqrt{M(q;x)}}\right)''\right]+U_{\alpha\beta}(x)$$

that, after using the potential $U_{\alpha\beta}(x)$
leads to:

(12)

given in Eq. (4)hand, if the domains D and D' match then the $\hat{p}_q$ operator could be self-adjoint. This latter property is also determined by the mass distribution $M(q;x)$.

Canonical Transformation

To solve the canonical Schrödinger equation $H\psi=E\psi$ with H given in Eq. (10) for some interaction potential V , we propose the point canonical transformation:

$$u_{eff}(x)=V(x)+\frac{\hbar^2}{4m_0}\left(\beta+\frac{1}{2}\right)\frac{M''(q;x)}{M^2(q;x)}$$
$$-\frac{\hbar^2}{2m_0}\left[\alpha(\alpha+\beta+1)+\beta+\frac{9}{16}\right]\frac{(M'(q;x))^2}{M^3(q;x)}$$

$x_q=\int$

$\sqrt{}$

$M(q;x)dx$(18)

(13)

The particular case of constant mass $m(0;x)=m_0$ gives place to the standard operators:
leading to:

$$-\frac{\hbar^2}{2m_0}\frac{d^2\psi}{dx_q^2}+\frac{\hbar^2}{2m_0}\left(ln\sqrt{m(q;x_q)}\right)'\frac{d\psi}{dx_q}+\left[V+\frac{\hbar^2}{2m_0}(\beta+1)\left(ln\sqrt{m(q;x_q)}\right)''\right.$$
$$\left.-\frac{\hbar^2}{2m_0}(4\alpha(\alpha+\beta+1)+\beta+1)\left(\left(ln\sqrt{m(q;x_q)}\right)'\right)^2\right]\psi=E\psi$$

(19)

$\hat{p}_0=\hat{p}=-i\hbar d/dx$ and $u_{eff}(x)=V(x)$ (14)

It is worth mentioning that the generalized linear

Thus, by applying the similarity transformation:

$\psi = \left(m(q;x)\right)^{1/4}\phi(x\ )$

q

(20)

momentum operator $\hat{p}_q$ given in Eq. (11) is a Hermitian

we have:

operator. In fact, any operator of the form: $-\frac{\hbar}{2m}\frac{d^2\phi}{dx^2} + U\phi = E\phi$

(21)

$\hat{A} = -ig(x)\frac{}{dx}$

$-if\ (x)$

$0 \quad q$

where:

fulfill the condition:

$U = V +$

$\left(\beta + \frac{1}{}\right)\left(ln\ \ m(q;x)\right)'' 2m \quad 2$ $\frac{\hbar^2}{} \quad - \quad \sqrt{\quad}$

$\int(\hat{A}\psi)^*\psi\, dx) = \int\psi^*\hat{A}\psi\, dx) + i\int\psi^*(2f(x) - g'(x))\psi\, dx$ (16)

0 $\sqrt{\quad}$

$\frac{3}{}\ '^2$ $\frac{\hbar}{}\left(4\alpha\,(\alpha \quad - \ (\quad)\right.$

(22)

Here,

$\hat{p}_q = \hat{A}$

on condition to have

$g(x) = 2m_0(+\beta+1) + \beta + \ ln\ \ m(q;x)\ \ 4 \quad \frac{\hbar}{\sqrt{M(q;x)}}$

$f(x) = \frac{1}{2}\left(\frac{\hbar}{\sqrt{M(q;x)}}\right)'$

. In such case the second term of Eq.The Schrödinger equation given above has been solved for different mass distributions under different interaction potentials, so in this study, we will only focus on using the transformation derived in the previous

(16) banishes and we have:

$\int(\hat{p}_q\psi)^*\psi dx) = \int\psi^*\hat{p}_q\psi\, dx)$

showing that the operator

(7)

$\hat{p}_q$ is a hermitian operator
formalism to extend its applications in the q-deformed thermodynamic properties.

Generalization (q-Deformed) of Thermodynamic Properties

and consequently the Hamiltonian operator $\hat{H}$

given in
The q-Deformed Quantum Dynamic Variables

Eq. (10) is also Hermitian, which is a sufficient condition to deal with real eigenvalues. Additionally,
The generalized quantum dynamic variables, namely the q-deformed linear momentum operator p_q and the

canonical transformation x_q given in Eqs. (11) and (18) respectively, preserve invariant the quantum commutationAt this point, it is worth mentioning that the q-deformed exponential function $exp_q\,(x)$ given in Eq. (24) and the q-
relationship, namely $\lfloor\lceil x_q, p_q\rceil\rfloor = i\hbar$

Explicitly:

deformed logarithm function of Eq. (26) can be used to introduce a
generalized (q-deformed) partition function Z_q and consequently the internal energy U_q. Namely:

$$\left[\int\sqrt{M(q;x)}\,dx\ ,\ -\frac{i\hbar}{\sqrt{M(q;x)}}\frac{d}{dx}-\frac{i\hbar}{2}\left(\frac{1}{\sqrt{M(q;x)}}\right)'\right]=$$
$$-\left(\int\sqrt{M(q;x)}\,dx\right)\left(-\frac{i\hbar}{\sqrt{M(q;x)}}\frac{d}{dx}\right)-\left(\int\sqrt{M(q;x)}\,dx\right)\left(\frac{i\hbar}{2}\left(\frac{1}{\sqrt{M(q;x)}}\right)'\right)$$
$$+\frac{i\hbar}{\sqrt{M(q;x)}}\frac{d}{dx}\left(\int\sqrt{M(q;x)}\,dx\right)+\frac{i\hbar}{2}\left(\frac{1}{\sqrt{M(q;x)}}\right)'\left(\int\sqrt{M(q;x)}\,dx\right)$$
$$=\frac{i\hbar}{\sqrt{M(q;x)}}\frac{d\int\sqrt{M(q;x)}\,dx}{dx}=i\hbar$$

Ω

$$Z_q = \sum_i exp_q(-\beta \in_i)$$

(30)

and:

(23)

$U \;=-\; \partial \; ln(Z\;)$ ___(31)

Furthermore, through these new quantum dynamic
variables, we can introduce the generalized (q-deformed) exponential function:
$_q$ $\partial\beta$ $_q$

with β = 1 ___ kT

. Thence, due to the fact that the partition
$exp_q(x) = exp(x_q)$

such that:
$\lim exp_q(x) = exp(x)$(24)25)Function Z and the internal energy U are involved with other statistic potentials (Peña *et al*., 2016) through the so- called Legendre transformations, namely the Entropy $S = kln(Z) + k\beta U$, the Helmholtz free energy
$F = - \frac{1}{\ } ln(Z)$, and the heat capacity $C = -k\beta^2 \frac{\partial U}{\ }$, by

$q\to 0$ β ∂β

being $exp(x)$ the standard exponential function.

preserving the structure of the Legendre transformations, it is possible to get their corresponding generalized expressions as follow:$S_q = k\,ln_q(Z_q) + k\beta U_q$In addition, if the transformation given in Eq. (18) has an inverse, the generalized (q-deformed) logarithmic function will be: — ——

$F\ = -\frac{1}{}\ ln\ (Z\ )$ and $C\ = -k\beta^2 \frac{dU_q}{}$ and. The next section$_q$ β $_q$ $_q$ $_q$ $d\beta$

$n_q(x) = x_q^{-1}(ln(x))$ (26)

Generalized (q-Deformed) Statistic Properties

will give some explicit examples.

Application to Thermodynamic Properties

This section is devoted to showing the usefulness of the proposal by considering two different position-

It is well known that statistical properties such as the internal energy U, entropy S, free energy F, and heat capacity C are defined through the partition function $Z(T)$ and its logarithm $ln(Z)$ (Peña *et al.*, 2016). It is defined as:

dependent

mass distributions of hyperbolic type.

Mass Distribution m(q; x)=m₀ cosh²(qx)

In this case $M\ (q;\ x) = \cosh^2(qx)$ such that, from Eqs. (11) and (18) The q-deformed quantum dynamic

$Z = \sum^{\Omega} exp\left(-\in\ /kT\right)$

(27)

variables are:

$_i$ $_i$

$\hbar$ $\_\hat{p} = -i\ sech(qx)\ ^{d}\ + i\,sech(qx)tanh(qx)$(32)

where, Ω is the total number of allowed states of the system with probabilities given by the Boltzmann distribution (Tsallis, 2009):$_q$

and:dx 2

$p_i = \frac{1}{Z}$

$-exp\left(-\in_i /kT\right)$ _

(28)$x\ = \frac{1}{}sinh(qx)$(33)

on condition that, $\sum^{\Omega} p\ = 1$

Straightforwardly, for this case, the quantum commutation relation between x_q and p_q remains unchanged.

$_i$ $_i$

Furthermore, in accordance with Eq. (24), the

hus, the internal energy U comes from:

$U = \sum^{\Omega} p\ \in\ = -\frac{\partial}{}\ lnZ$(29) $exp_q(x) = exp\left(\frac{1}{q}sinh(qx)\right)$ generalized q-deformed exponential function is:

$_{i} \quad _{i} \quad _{i} \quad \partial\beta$(34)

whose partner inverse function is defined as the generalized q-deformed logarithmic relationship:and:

$C = -k\beta^{2}\frac{\partial U_q}{} = -k\beta^{2}\frac{\partial}{}\left(-\frac{\partial}{}ln\ (Z\ )\right) = \quad q \qquad 2\beta\partial\beta$

$n_q(x) = \frac{sinh(qln(x))}{2q}$(35)

$-$

$k\beta^{2}\ \partial\ \ ln\left(qln(Z\ ) + \_\_\_\right.$

$1 + q^{2}ln^{2}(Z\ )\ ^{\bar{q}}$.(45)

$\sqrt{\quad}\ )$ $\quad ^{1}$

So, by using the

identity: $\partial\beta^{2}$ $\quad _{q} \quad _{q}$

$sinh^{-1}(x) = ln\left(x + \sqrt{\quad}\right.$

one gets:

$ln\ (x) = ln\left\lceil qln(x) + x^{2} + 1\right)$

$1 + q^{2}\, ln^{2}(x)\ \rceil^{q}$ $\quad \sqrt{\quad}\ -$ $\quad ^{1}$

(36)

As before, when the q parameter tends to zero, all the above-generalized statistic properties reduce to their corresponding standard ones.

Mass Distribution m(q; x)=m₀ cosh⁴(qx)

In this new situation, from Eqs. (11) and (18) the q-

$\lfloor \qquad \rfloor$

Also, it is easily observed that:

deformed quantum variables are:

$\lim_{q\to 0} exp_q(x) = exp(x)$ and $\lim_{q\to 0} ln_q(x) = ln(x)$ $\quad dx \quad \hbar$ (38) $\hat{p}_q = -i\hbar\ cosh^{2}(qx)\frac{d}{} - i\ qcosh(qx)sinh(qx)$

(46)

Consequently, from the Eqs. (30) and (34) we can write the q-deformed partition function $Z_q\ (T)$ as:
and:

$x = \frac{1}{q}tanh(qx)$ $\quad -$

(47)

$Z\sum_{i}\ ^{\Omega} exp\left(\frac{-sinh(q\beta \in_i)}{q}\right)$(39)fulfilling the commutation relationship $[x_q, p_q] = ih$

Hence, the generalized (q-deformed) internal energy comes from the Eq. (31) as:
Likewise, from Eq. (24), the generalized q-deformed exponential function results in:

$-$ $\quad xp\ (x) = exp\left(\frac{1}{}tanh(qx)\right)$(48)

$$U_q = -\frac{\partial}{\partial\beta}ln\left(q\, ln(Z_q) + \sqrt{1 + q^{2}ln^{2}(Z_q)}\right)^{\frac{1}{q}}$$

Also, in view of Eq. (38), one has:

(40)⎝ ⎠and the corresponding inverse function is:

$ln\ (x) = \frac{1}{\ } tanh^{-1}(qln(x))$

(49)

$\lim_{q\to 0} Z_q = Z$

and:(41)$_q$ q

which, after using the identity: $tanh^{-1}(x) = \frac{1}{\ } ln\left(1+x\right)$(50)

$\lim U_q = -\frac{\partial}{\ }$ − ⎝ ⎠ $\to 0$ $\partial\beta ln(Z_q) = U$(42)2 | $1-x$

we have:

Finally, by following the structure of the Legendre transformation among some thermodynamic functions, the generalization of the internal en $ln_q(x) = \frac{1}{2q} ln\left(\frac{1+qln(x)}{1-qln(x)}\right)$ and its related functions are rewritten as follows:

(51)

$S_q = k\,ln_q(Z_q) - k\beta U_q$(43)Also, the $q\longrightarrow 0$ limit leads to:

$\lim exp_q(x) = exp(x)$

(52)

where:

$$U_q = \frac{\partial}{\partial\beta} ln\left(q\, ln(Z_q) + \sqrt{1 + q^2 ln^2(Z_q)}\right)^{\frac{1}{q}},$$

$$F_q = -\frac{1}{\beta} ln\left(q\, ln(Z_q) + \sqrt{1 + q^2 ln^2(Z_q)}\right)^{\frac{1}{q}}$$

(44)
$q\to 0$

and:

$\lim ln\ (x) = ln(x)_{\to 0}$ q

(53)

Eq. (30) together with Eq. (48) leads to the q-deformed partition function:

$$Z_q = \sum_i^{\Omega} exp\left(\frac{1}{q} tanh(-q\beta\epsilon_i)\right) \qquad (54)$$

Hence, the generalized (q-deformed) internal energy comes from the Eq. (31) as:1 Materials and Methods

This study is theoretical research on the field of Quantum mechanics. Specifically, on the q-deformed form for which the study begins with the search of the q- deformed Quantum linear momentum operator. From there, the q-deformed exponential function $exp_q(x) = exp(x_q)$ as well as the corresponding q-deformed logarithm function $ln_q(x) = x_q^{-1}(ln\,(x))$, which come from the canonical form of the PDM Schrödinger equation, were used to generalize the thermodynamic potentials

$$\partial_q =- \Big(1 + qln(Z_q) \Big)^{2q}$$

$$ln\Big| \quad \overline{\partial\beta} \quad \Big(1 - qln(Z_q) \Big)$$

By virtue of the result given in Eqs. (52-53) we have:such as $m(q;x) = m_0cosh^2(qx)$ and $m(q;x) = m_0cosh^4(qx)$ were used for exemplifying the proposal.

$$\operatorname*{im}_{q\to 0} Z_q = Z$$

as well as:

$$\lim U_q = U_{q\to 0}(56)$$

(57)

Results and Discussion

From the hyperbolic mass distribution $m(q;x) = m_0cosh^2(qx)$, the q-exponential function:

$exp_q(x) = exp\left(\frac{1}{q}sinh(qx)\right)$Finally, the Legendre transformations of the thermodynamic functions are generalized as follows:and the q-logarithm function:

$$ln_q(x) = ln\left[qln(x) + \sqrt{1 + q^2ln^2(x)}\right]^{\frac{1}{q}}$$

$$S_q = k\,ln_q(z_q) - k\beta U_q$$

where:

$$U_q = \frac{1}{2q}\frac{\partial}{\partial\beta}ln\left(\frac{1 + qln(Z_q)}{1 - qln(Z_q)}\right)$$

$$F_q = -\frac{1}{\beta}ln_q(Z_q) = -\frac{1}{2\beta q}ln\left(\frac{1 + qln(Z_q)}{1 - qln(Z_q)}\right)$$

and:

$$C_q = -k\beta^2\frac{\partial\,U_q}{\partial\beta} = \frac{k\beta^2}{2q}\frac{\partial^2}{\partial\beta^2}ln\left(\frac{1 + qln(Z_q)}{1 - qln(Z_q)}\right)(58)$$

(59)

(60)
(61)
were obtained.

With these results, the corresponding generalized thermodynamic properties become:

$$U_q = -\frac{\partial}{\partial\beta} ln\left(q\, ln(Z_q) + \sqrt{1 + q^2 ln^2(Z_q)}\right)^{\frac{1}{q}}$$

$$S_q = k\, ln_q(Z_q) - k\beta U_q$$

$$F_q = -\frac{1}{\beta} ln\left(q\, ln(Z_q) + \sqrt{1 + q^2 ln^2(Z_q)}\right)^{\frac{1}{q}}$$

and:

$$C_q = k\beta^2 \frac{\partial^2}{\partial\beta^2} ln\left(q\, ln(Z_q) + \sqrt{1 + q^2 ln^2(Z_q)}\right)^{\frac{1}{q}}$$

As expected, from expressions given in Eqs. (56-57) the limit case $q \to 0$ leads to the standard statistic potentials, namely:

$$\lim_{q\to 0} S_q = S,\ \lim_{q\to 0} F_q = F\ and\ \lim_{q\to 0} C_q = C \qquad (62)$$

Likewise, when we use the mass distribution $m(q;x) = m_0 cosh^4(qx)$, the q-exponential function and the q-logarithm functionare respectively given by:

Recovering the standard expressions for the statistic properties entropy S, Helmholtz free energy F and the heat capacity C.

$$exp_q(x) = exp\left(\frac{1}{q} tanh(qx)\right)$$

and:

$$ln_q(x) = \frac{1}{2q} ln\left(\frac{1+qln(x)}{1-qln(x)}\right)$$

Consequently, the corresponding generalized thermodynamic properties:

$$U_q = \frac{1}{2q}\frac{\partial}{\partial\beta} ln\left(\frac{1+qln(Z_q)}{1-qln(Z_q)}\right)$$

$$F_q = -\frac{1}{\beta} ln_q(Z_q) = -\frac{1}{2\beta q} ln\left(\frac{1+qln(Z_q)}{1-qln(Z_q)}\right)$$

and:

$$C_q = -k\beta^2 \frac{\partial U_q}{\partial\beta} = \frac{k\beta^2}{2q}\frac{\partial^2}{\partial\beta^2} ln\left(\frac{1+qln(Z_q)}{1-qln(Z_q)}\right)$$

were derived.

Conclusion

The goal of this research has been to use the q-deformed quantum variables xq and pq derived from an algebraic method to suggest a generalization of the partition function and the internal energy by means of a q-deformed exponential function and its companion q-deformed logarithmic function. Furthermore, by using the aforementioned method, some thermodynamic characteristics may be found in the generalized partition function and internal energy. These generalizations are simple because the Legendre relations, which are treated as invariants, are used to express these potentials. In particular, all other related thermodynamic functions are directly generalized from the q-deformed generalized partition function Z = exp(ℰぢ). Although the technique is generalizable, we have only examined two hyperbolic position-dependent mass distributions to demonstrate the practicality of our idea.

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