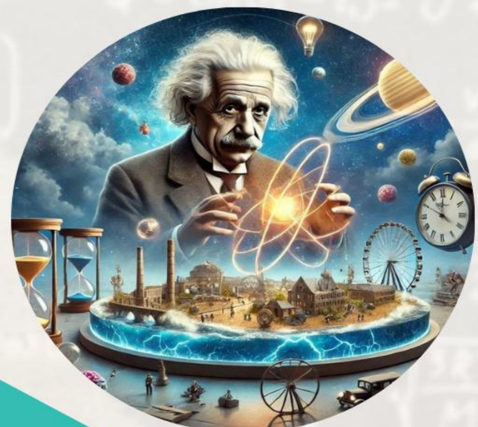
MULTISCIA JOURNALS PUBLISHERS

FRONTIERS IN MATHEMATICS AND BIOSTATISTICS

ISSN: (3065- 4297)

https://multisciajournals.com/journals/index.php/fmbs

editor.fmbs@gmail.com



Frontiers in Mathematics and Biostatistics
Volume1, Issue1, 2025

Statistical Inference for Quantiles of Two-Parameter Gamma Distribution: A Generalized Approach

Malwane Ananda

Department of Maths

Article Info

Received: 22-12-2024 Revised:21-01-2025 Accepted:05-02-2025 Published:15-02-2025

Abstract: Hydrology, meteorology, environmental monitoring, longevity testing, and dependability are just a few of the numerous fields that make use of the gamma distribution for data analysis. The quantiles of a two-parameter gamma distribution are the focus of this statistical investigation. Particularly in domains like life testing and flood frequency analysis, gamma quantile testing and estimate are necessary. Statistical inference methods included in published works on the subject are all approximation techniques. Our research suggests two approaches to this issue. An accurate statistical inference strategy based on the generalized p-value methodology is the first approach. The method relies on precise probability assertions instead of approximations, making it accurate. The second method follows the same principle as the parametric bootstrap method. We evaluate the suggested approaches using various real-world data sets and contrast the outcomes with those of competing approaches. To evaluate how well the suggested approaches work in comparison to other ways that are already out there, a brief simulation study is provided. Whether we're talking about lower or higher quantiles, the simulation results show that these two new approaches outperform the other current ones in terms of size and power.

Keywords: Generalized Inference, Generalized p-values, Gamma Quantiles, Confidence Limits, Tolerance Limits

Introduction

When analyzing data on rainfall, pollutants, environmental monitoring, and lifetimes, the gamma distribution is a popular choice. For hydrological and rainfall data analysis, gamma distributions (Aksoy, 2000; Ashkar and Bobèe, 1988; Ashkar and Ouarda, 1998), such the log-pearson type 3 and the Pearson type 3, are used.

Let X be a subset of $G(\alpha, \beta)$, where O and β are parameters that are not known. Both the shape parameter (O) and the scale parameter (β) are used here. The qth quantile, denoted as θq, is the relevant parameter. It is defined as $Pr (X \leq \theta q) = q$, with $0< q <1$ being a specified constant. Making use of a random sample X1, X2,..., Xn, we want to establish an LCL and UCL and then test the UCL or lower-tailed hypothesis: standard deviations are used often. The gamma model is a useful tool for analyzing exposure and pollution data. competing models with lognormal ones (Maxim et al., 2006). Hypothesis 0: $\theta < \delta$ in opposition to Hypothesis 1: $\theta q < B$ (1)

The use of groundwater and environmental monitoring data is shown by Gibbons (1995). The gamma distribution is an excellent fit for rainfall data models, as Husak et al. (2007) shown.
It is common practice in these fields to estimate gamma distribution quantiles, tolerance limits, and prediction limits. The quantiles of a two-parameter gamma distribution are the focus of statistical inference in this piece. where the constant δ is known.
There are a plethora of articles in the current literature that discuss various aspects of gamma parameter testing and estimation. The vast majority of statistical inference processes found in published works are approximations when both parameters are unknown. The most powerful test for testing the scale parameter β was provided by Engelhardt and Bain (1977). An approximation test for evaluating the gamma was developed by Bain et al. (1984).

mean $\alpha\beta$. Bhaumik *et al.* (2009) provided tests for the shape parameter α and the mean $\alpha\beta$. Krishnamoorthy and León-Novelo (2014) provided approximate small sample likelihood-based statistical inference procedures for the gamma parameters and the mean.

Bain *et al.* (1984) proposed an approximate tolerance limit for the two-parameter gamma distribution. Their method consists of replacing the unknown shape parameter α with the maximum likelihood estimate of it and utilizing an approximation that is accurate only for small values of q of the gamma distribution. They showed that their method provides satisfactory estimates for the LCL when the value of q is lower than 0.2. Their method works well with their applications since the lower tail is more important than the upper tail in reliability and industrial engineering applications. However, this is not the case with most applications that arise in hydrological applications and flood frequency analysis (Ashkar and Ouarda, 1998), the higher values of q are often the ones under scrutiny. Therefore, they are more interested in the values of θ_q for higher values of q. For example, in flood frequency analysis, the 100-year flood is associated with the quantile θ_q with $q = 0.99$; it is often desired to get LCL for θ_q (Ashkar and Bobée, 1988). Furthermore, these types of data are highly skewed and the gamma distribution and gamma-related distributions are frequently used to analyze the data in these areas. In these areas of research, there are many papers in the literature related to estimation θ_q; but they all use approximate methods.

Ashkar and Ouarda (1998) gave an approximate method to construct confidence limits for θ_q using transformed data. This method is frequently used in hydrology and flood frequency analysis to construct confidence intervals for θ_q. Their method is briefly described in the next section. By utilizing the $X^{1/3}$ transformation, Krishnamoorthy *et al.* (2008) proposed an approximate method for testing and estimating the parameters and functions of gamma parameters. Their method is also briefly described in the next section.

Aryal *et al.* (2008) showed that the two-parameter gamma distribution can be approximated by a normal distribution when the shape parameter is large. They suggested using normal-based tolerance limits if the maximum likelihood estimator of the α is more than 7. Otherwise, they provided tabular values to construct tolerance factors.

Weerahandi and Gamage (2016) introduced a general method to tackle statistical inference problems for a two-parameter continuous distribution. Using this method, they showed that one can get an exact test and confidence intervals for the gamma parameters and the mean using distribution. The test is exact in the sense that it is based on exact probability statements rather than based on approximations. Furthermore, we introduce a parametric bootstrap method to carry out statistical inference of θ_q.

Inference for Gamma Quantiles

The minimal sufficient statistics for α and β are given by $S = \Sigma_{i=1}^{n} X_i$ and $P = \Pi_{i=1}^{n} X_i$. These two statistics S and P are not independent, but a statistic T defined as a ratio of the geometric mean and the arithmetic mean is independent of S (Bhaumik *et al.*, 2009). Therefore, the joint statistic (S, T) is defined by:

$$S = \sum_{i=1}^{n} X_i \quad and \quad T = \frac{\left(\Pi_{i=1}^{n} X_i\right)^{1/n}}{\left(\Sigma_{i=1}^{n} X_i / n\right)/n} = \frac{P^{1/n}}{S/n} \tag{2}$$

provides an independently distributed joint sufficient statistic for the statistical inference regarding parameters α and β. In the first two subsections, we summarize two leading statistical procedures and the confidence bounds for the gamma quantiles.

Materials and Methods

An approximation method for building the confidence limits from modified data was proposed by Ashkar and Ouarda (1988). This technique is called the AB approach. With X being a gamma random variable with the cdf FX and Y being a normal random variable with the cdf FY, their approximation process is predicated on the assumption that X is distributed as FX-1 (FY (Y)). They calculated the following upper limit and lower limit for θq, which are roughly 100(1-γ)%.

$$LCL = \bar{X} + S_X K_{p1}, \quad UCL = \bar{X} + S_X K_{p2} \tag{3}$$

where, $\bar{X} = \Sigma_{i=1}^{n} X_i / n$, $S_X^2 = \Sigma_{i=1}^{n} (X_i - \bar{X})^2 / (n-1)$; K_{p1} and K_{p2} are the values of p_1^{th} and p_2^{th} quantiles of the standardized gamma variate with the coefficient of skewness υ. The coefficient of skewness for the gamma distribution $\upsilon = 2/\sqrt{\alpha}$ is estimated using the maximum likelihood estimate of α. The values of p_1 and p_2 are given by:

$$p_1 = \Phi\left(\frac{1}{\sqrt{n}} t_{n-1,\gamma} \left(z_q \sqrt{n}\right)\right)$$

and:

the generalized p-value approach. In this article, we utilize their approach to find an exact

Here Φ (.) is the standard normal cdf, z_q is the q^{th} quantile of the standard normal distribution and $t_{a;b}(c)$ is the b^{th} quantile of the noncentral t-distribution with degrees of freedom a and noncentrality parameter c. A two-sided approximate 100 (1-2γ)% confidence interval for θ_q is given by (LCL, UCL), where LCL and UCL are given in (3).

The values of K_p can be approximated using the formula:

$$K_p \approx \frac{2}{\upsilon}\left\{\left[\frac{\upsilon}{6}\left(z_p - \frac{\upsilon}{6}\right)+1\right]^3 - 1\right\}$$

where, $\upsilon = 2/\sqrt{\alpha}$ is the coefficient of skewness of the gamma distribution, which is estimated using the maximum likelihood estimate. This approximation works well when $\upsilon < 2$ and many other approximations are given for other values. The references for these approximations and tabulated values for K_p are given in Bobee and Ashkar (1991).

Normal Approximation Approach

Aryal *et al*. (2008) argued that the gamma distribution can be approximated by a normal distribution when the maximum likelihood estimator of the shape parameter of the gamma distribution is larger than 7. Krishnamoorthy *et al*. (2008) pointed out that the normal approximation appears to not be useful for statistical inference related to gamma parameters, but it is accurate for the computation of the prediction intervals, tolerance intervals and for statistical inference regarding stress-strength reliability parameters. Krishnamoorthy *et al*. (2008) investigated the accuracy and appropriateness of the $X^{1/3}$ transformation suggested by Wilson and Hilferty (1931) to analyze the gamma data. They showed that the $X^{1/3}$ transformation works much better than the $X^{1/4}$ transformation which is known as the Hawkins and Wixley (1986) approximation. After transforming the data, they utilized the available normal tolerance intervals (Guttman, 1967) to carry out the inferences for the gamma data. Overall, they concluded that the $X^{1/3}$ transformation provides a simple, easy-to-use approach for dealing with various problems related to the gamma distribution. In our analysis, we use the $X^{1/3}$ transformation approach to analyze the data. From here on we will refer to the method associated with the $X^{1/3}$ transformation as the Normal Approximation (NA) method.

With the normal approximation method, approximate 100 (1-γ)% LCL and UCL for the gamma quantile θ_q is given by:

$$LCL = \left[\bar{Y} + c_1 S_Y\right]^3, \quad UCL = \left[\bar{Y} + c_2 S_Y\right]^3 \qquad (4)$$

where, $Y = X^{1/3}, i = 1, 2, \ldots, n; \bar{Y} = \Sigma^n Y / n, S^2 = \Sigma^n (Y - \bar{Y})^2 / (n-1)$

where, z_q is the q^{th} quantile of the standard normal distribution and $t_{a,b}(d)$ is the b^{th} quantile of the noncentral t-distribution with degrees of freedom a and noncentrality parameter d. The two quantities c_1 and c_2 are referred to as tolerance factors. The two-sided approximate 100 (1-2γ)% confident interval for θ_q is given by (*LCL*, *UCL*), where *LCL* and *UCL* are given in (4).

Statistical inference for the tolerance limits and confidence intervals go hand in hand. To describe this, let q denote the "content" and 1-γ denote the coverage robability of a tolerance interval. Then the Upper Tolerance Limit (UTL) for the two-parameter gamma distribution with content q and the coverage probability 1-γ is given by the *UCL* that is given in 4.

The Lower Tolerance Limit (LTL) with content q and the coverage probability 1-γ can be obtained by replacing the plus sign with the minus sign in the *UCL* formula, i.e., $LTL = [\bar{Y} + c_2 S_Y]^3$. Or else, *LTL* can be obtained by computing 100 (1-γ)% *LCL* given in (4) for the $(1\text{-}Q)^{th}$ quantile $\theta_{1\text{-}q}$.

Generalized Inference Procedure

To describe the method introduced by Weerahandi and Gamage (2016), let $X_1, X_2, \ldots, X_n$ be a random sample from a two-parameter continuous density $X \sim f(x;\alpha,\beta)$ with a joint minimal sufficient statistics (S, T). Both parameters are assumed to be unknown and let be β the parameter of interest. Let $F_T(t; \alpha, \beta) = Pr(T \le t)$ be the cdf of T. Then $U = F_T(T)$ has a uniform distribution over 0 and 1 and $U = U(T;\alpha,\beta)$ is a function of α,β and T. At the observed value t of T, if $u(t)$ is the value of u, then $u(t) = F_t(t; \alpha, \beta)$ For a fixed value of t, this is a function of α and when treated as a function of α, $u(\alpha) = F_T(t; \alpha, \beta)$. Let u^{-1} be its inverse function satisfying the equation $u^{-1}(u(\alpha)) = \alpha$. Now, define the random variable $R_b(T; \alpha, \beta, t) = u^{-1}(U(T))$, which satisfies the following two properties:

1. At the observed value t of T, $R_b(t;\alpha, \beta, t) = \alpha$ and
2. The distribution of R_b is free of the parameter α

The random quantity $R_b(T; \alpha, \beta, t)$, at times, is denoted as $\hat{\alpha}(U)$ since this is the quantity that is used to replace α the construction of the generalized pivotal quantity.

Now, let $F_{S/T=t}(s;\alpha,\beta)$ be the cdf of the conditional distribution of S given $T = t$. Let us denote it by $W(S, t) = F_{S/T=t}(S)$ and it is distributed as $W \sim U(0, 1)$. This Therefore, the unconditional distribution of $W(S, T)$ is $U(0, 1)$ and

Here c_1 and c_2 are given by:

it is independent of T.

For the testing and estimation β, Weerahandi and Gamage (2016) defined a Generalized Test Variable

$c = \frac{1}{\sqrt{n}}$

$= \frac{W(S,T;\alpha,\beta)}{w(s,t;\hat{\alpha}(U),\beta)}$ $\sqrt{n}$

where, W and U are independent uniform $U \sim U(0, 1)$ random variables.

If the cdf of R is monotonically increasing with respect to the parameter of interest β, then the hypothesis $H_0 : \beta \le \beta_0$ $H_1 : \beta > \beta_0$ can be tested using the p-value:

$$p = Pr\left(\frac{W(S,T;\alpha,\beta)}{w(s,t;\hat{\alpha}(U),\beta_0)} \le 1 \right) = E\left(w(s,t;\hat{\alpha}(U),\beta_0)\right)$$

atisfying the equation $u^{-1}(u(\alpha)) = \alpha$. Now, define the random variable $R_b(T;\alpha,t) = u^{-1}(u(T))$, which satisfies the following two properties:

1. At the observed value t of T, $R_b(t;\alpha,t) = \alpha$
2. The distribution of R_b is free of the parameter α

The quantity $R_b(T;\alpha,t)$ will be used to replace α the construction of the generalized pivotal quantity and therefore let us denote it by $\hat{\alpha}(U) = R_b(T;\alpha,t)$.

Now consider the cdf of the conditional distribution of

where, the expected value is taken with respect to the uniform $U(0, 1)$ random variable.

Generalized Inference Procedure for Gamma Quantiles

S given $T = t$, $F_{S/T=t}(s;\alpha,\theta_q)$ .As before, if we denote it by $F_{S/T=t}(S)$, W is distributed as $U(0,1)$. Since S and T are independent, $W(S, T)$ and T are independent. Now we can de ne a generalized pivotal quantity for as:

Now let us consider the statistical inference for the q^{th}quantile θ_q of the gamma distribution $G(\alpha, \beta)$; where both α

$$R = \frac{W}{W(S,T;\alpha,\theta_q)} = \frac{w(s,t;R_b(T;\alpha,t),\theta_q)W}{w(s,t;\hat{\alpha}(U),\theta_q)}$$

and β are unknown parameters. Here α is the shape parameter and β the scale parameter. The joint statistics (S, T) defined in (2) is an independently distributed complete minimal sufficient statistic for α and β .

Let $F_{\alpha,\beta}(.)$ Denote the cdf of the gamma $G(\alpha, \beta)$ distribution. Then $F_{\alpha,\beta}(\theta_q) = q$ and:

$\theta_q/\beta \; y^{\alpha-1}e^{-y}$

where, $W = W(S,T;\alpha,\theta_q)$ and $U = U(T)$ are independent uniform $U(0, 1)$ random variables. The value of R is equal to one at the observed values s of S and t of T. Furthermore, the distribution of R is free of the nuisance parameter α *and* therefore R is a generalized pivotal quantity for θ_q.

The cdf of the R is monotonically non-decreasing with respect to the parameter θ_q and the hypothesis $H_0 : \theta_q \le \delta$

$$\int_0 \frac{}{\Gamma(\alpha)} dy = q$$

(5)

against $H_1: \theta > \delta$ can be tested using the p-value.

The parameter of interest $\theta_q = \theta_q(\alpha, \beta)$ can be written as:

$$p = Pr\left(\frac{W(S,T;\alpha,\theta_q)}{} \le 1 \right) = E\left(w(s,t;\hat{\alpha}(U),\delta)\right) \Big| \; w(s,t;\hat{\alpha}(U),\delta$$

$$\theta(\alpha,\beta) = \beta F^{-1}(q) = \beta\theta(\alpha)$$

Table 1: Type I error performance of competing tests $\beta = 1$

The data set was originally reported by Gibson (1994) and the gamma distribution was shown to provide a good fit to this data. Aryal *et al*. (2008) used this data to demonstrate the appropriateness of the straightforward normal approximation if the shape parameter is larger than 7. The 90% confidence intervals for $\theta_{.9}$ using the AB, NA, GM and PB methods yield (74.780, 96.857), (75.084, 97.705), (75.029, 97.812) and (72.664, 93.178) respectively.

The 90% confidence intervals for $\theta_{.1}$ using AB, NA, GM and PB methods yield (0.092, 0.384), (0.067, 0.379). (0.084, 0.378) and (0.114, 0.430) respectively. Note that the Upper Tolerance Limits (UTL) with a content of .95 and coverage probability of 0.9 using the methods AB, NA, GM and PB are given by 96.875, 97.705, 97.812 and 93.178, respectively. Similarly, the Lower Tolerance Limits (LTL) with a content of .95 and coverage probability of 0.9 using the methods AB, NA, GM and PB are given by 28.805, 28.343, 28.180 and 30.376 respectively.

Results

Here we provide a small-scale simulation research that compares the AB, NA, GM, and PB approaches. We will examine the lower and higher assumptions given in (1) by evaluating the size performance of these strategies in the remainder of the simulation research. Here are the settings for parameters and sample size that we use: P = 0.5, 1.0, 1.5, 5.0, β = 1.0, 5.0, with n = 10, 20, and n+1. For quantile values q = 0.1, 0.3, 0.5, 0.7, and 0.9, 5000 simulated samples were used to test the lower and higher hypotheses given in (1) for each of these configurations. The target level is established at 0.05 for all these size estimates. As shown in Tables 1-2, the real sizes, or the simulated rejection rates, were computed. The GM and PB approaches use simulations for their computations, as described in chapter 2, and the simulated sample sizes for both methods are maintained at 5000.

Actual size of lower/upper-tailed tests for testing q^{th} quantile (intended level 0.05)

q:	0.1	0.3	0.5	0.7	0.9
Method/case:		$\alpha = 0.5, \beta = 1, n = 10$			
GM	0.048(0.043)	0.049(0.041),	0.060(0.040)	0.060(0.040)	0.055(0.040)
NA	0.019(0.025)	0.019(0.053)	0.042(0.062)	0.066(0.058)	0.082(0.044)
AB	0.037(0.065)	0.040(0.063)	0.050(0.055)	0.067(0.043)	0.088(0.034)
PB	0.055(0.053)	0.054(0.052)	0.051(0.052)	0.051(0.051)	0.050(0.054)
Method/case:		$\alpha = 1.0, \beta = 1, n = 10$			
GM	0.031(0.036)	0.055(0.036)	0.058(0.026)	0.044(0.041)	0.055(0.038)
NA	0.033(0.042)	0.036(0.051)	0.047(0.053)	0.057(0.051)	0.059(0.046)
AB	0.037(0.067)	0.043(0.064)	0.053(0.056)	0.066(0.046)	0.076(0.038)

Table 1: Continue

PB	0.055(0.053)	0.052(0.050)	0.048(0.050)	0.049(0.052)	0.050(0.053)
Method/case:		$\alpha = 1.5, \beta = 1, n = 10$			
GM	0.046(0.042)	0.051(0.042)	0.059(0.045)	0.060(0.046)	0.059(0.042)
NA	0.039(0.045)	0.043(0.050)	0.049(0.051)	0.054(0.050)	0.054(0.047)
AB	0.038(0.068)	0.047(0.064)	0.056(0.057)	0.065(0.047)	0.071(0.038)
PB	0.056(0.052)	0.050(0.051)	0.049(0.053)	0.051(0.052)	0.052(0.053)
Method/case:		$\alpha = 5, \beta = 1, n = 10$			
GM	0.036(0.036)	0.046(0.048)	0.054(0.064)	0.046(0.047)	0.046(0.025)
NA	0.047(0.049)	0.048(0.051)	0.050(0.050)	0.051(0.050)	0.051(0.049)
AB	0.039(0.068)	0.049(0.063)	0.058(0.058)	0.063(0.049)	0.068(0.039)
PB	0.051(0.051)	0.051(0.051)	0.047(0.051)	0.051(0.052)	0.050(0.055)
Method/case:		$\alpha = 0.5, \beta = 1, n = 20$			
GM	0.051(0.054)	0.052(0.048)	0.059(0.044)	0.055(0.042)	0.054(0.037)
NA	0.020(0.022)	0.011(0.070)	0.030(0.079)	0.063(0.063)	0.097(0.038)
AB	0.040(0.061)	0.039(0.059)	0.044(0.049)	0.057(0.040)	0.078(0.038)
PB	0.054(0.051)	0.051(0.052)	0.053(0.053)	0.052(0.052)	0.050(0.050)
Method/case:		$\alpha = 1.0, \beta = 1, n = 20$			
GM	0.047(0.034)	0.047(0.034)	0.053(0.046)	0.055(0.046)	0.058(0.049)
NA	0.030(0.040)	0.030(0.057)	0.044(0.057)	0.057(0.052)	0.065(0.042)
AB	0.041(0.064)	0.043(0.059)	0.049(0.051)	0.058(0.045)	0.070(0.040)
PB	0.055(0.049)	0.050(0.049)	0.051(0.051)	0.052(0.052)	0.050(0.050)
Method/case:		$\alpha = 1.5, \beta = 1, n = 20$			
GM	0.052(0.042)	0.053(0.042)	0.062(0.043)	0.062(0.044)	0.054(0.044)
NA	0.036(0.046)	0.039(0.053)	0.046(0.053)	0.054(0.050)	0.057(0.045)
AB	0.041(0.063)	0.045(0.058)	0.051(0.053)	0.059(0.045)	0.067(0.041)
PB	0.054(0.050)	0.052(0.051)	0.052(0.052)	0.049(0.050)	0.051(0.051)
Method/case:		$\alpha = 5.0, \beta = 1, n = 20$			
GM	0.041(0.051)	0.042(0.048)	0.052(0.038)	0.052(0.044)	0.054(0.043)
NA	0.047(0.049)	0.048(0.050)	0.049(0.051)	0.052(0.050)	0.051(0.048)
AB	0.040(0.062)	0.047(0.058)	0.054(0.054)	0.057(0.048)	0.063(0.041)
PB	0.052(0.051)	0.052(0.053)	0.052(0.052)	0.052(0.052)	0.051(0.050)

Table 2: Type I error performance of competing tests when $\beta = 5$

Actual size of lower/upper-tailed tests for testing q^{th} quantile (intended level 0.05)

q:	0.1	0.3	0.5	0.7	0.9
Method/case:		$\alpha = 0.5, \beta = 5, n = 10$			
GM	0.049(0.044)	0.049(0.041)	0.060(0.042)	0.060(0.038)	0.053(0.042)
NA	0.019(0.026)	0.020(0.052)	0.041(0.061)	0.066(0.057)	0.083(0.044)
AB	0.036(0.066)	0.040(0.065)	0.051(0.056)	0.067(0.043)	0.088(0.035)
PB	0.054(0.053)	0.054(0.053)	0.047(0.052)	0.050(0.053)	0.050(0.053)
Method/case:		$\alpha = 1.0, \beta = 5, n = 10$			
GM	0.040(0.036)	0.057(0.045)	0.058(0.047)	0.061(0.040)	0.056(0.036)
NA	0.032(0.041)	0.036(0.051)	0.048(0.053)	0.056(0.051)	0.059(0.046)
AB	0.038(0.068)	0.044(0.064)	0.054(0.057)	0.065(0.046)	0.075(0.037)
PB	0.055(0.052)	0.052(0.051)	0.051(0.051)	0.052(0.051)	0.050(0.054)
Method/case:		$\alpha = 1.5, \beta = 5, n = 10$			
GM	0.045(0.043)	0.050(0.042)	0.056(0.047)	0.061(0.045)	0.059(0.040)
NA	0.038(0.046)	0.042(0.051)	0.048(0.052)	0.054(0.050)	0.054(0.048)
AB	0.038(0.068)	0.046(0.064)	0.056(0.056)	0.065(0.049)	0.072(0.038)
PB	0.052(0.048)	0.052(0.051)	0.053(0.050)	0.049(0.052)	0.049(0.053)
Method/case:		$\alpha = 5, \beta = 5, n = 10$			
GM	0.056(0.048)	0.064(0.050)	0.057(0.052)	0.053(0.049)	0.053(0.038)
NA	0.048(0.049)	0.049(0.050)	0.050(0.050)	0.050(0.049)	0.051(0.049)
AB	0.037(0.067)	0.049(0.064)	0.057(0.058)	0.063(0.049)	0.069(0.039)
PB	0.055(0.050)	0.054(0.049)	0.051(0.050)	0.051(0.052)	0.049(0.053)
Method/case:		$\alpha = 0.5, \beta = 5, n = 20$			

DOI: 10.3844/jmssp.2024.1.12

Table 2: Continue

GM	0.051(0.055)	0.051(0.048)	0.060(0.044)	0.055(0.042)	0.054(0.037)
NA	0.020(0.022)	0.011(0.069)	0.031(0.080)	0.064(0.063)	0.098(0.037)
AB	0.041(0.061)	0.039(0.058)	0.043(0.049)	0.056(0.040)	0.077(0.038)
PB	0.054(0.051)	0.054(0.052)	0.050(0.050)	0.047(0.052)	0.049(0.054)
Method/case:		$\alpha = 1.0, \beta = 5, n = 20$			
GM	0.048(0.044)	0.047(0.043)	0.054(0.041)	0.049(0.043)	0.055(0.047)
NA	0.030(0.041)	0.030(0.057)	0.044(0.058)	0.056(0.051)	0.064(0.043)
AB	0.040(0.063)	0.043(0.057)	0.049(0.052)	0.059(0.044)	0.069(0.040)
PB	0.052(0.051)	0.050(0.050)	0.052(0.052)	0.051(0.052)	0.050(0.052)
Method/case:		$\alpha = 1.5, \beta = 5, n = 20$			
GM	0.059(0.049)	0.056(0.047)	0.048(0.042)	0.048(0.043)	0.46(0.044)
NA	0.037(0.046)	0.038(0.053)	0.047(0.053)	0.054(0.050)	0.057(0.045)
AB	0.040(0.063)	0.045(0.059)	0.050(0.052)	0.058(0.045)	0.066(0.041)
PB	0.051(0.051)	0.052(0.051)	0.050(0.050)	0.049(0.052)	0.051(0.051)
Method/case:		$\alpha = 5.0, \beta = 5, n = 20$			
GM	0.045(0.048)	0.053(0.058)	0.057(0.061)	0.052(0.054)	0.053(0.057)
NA	0.047(0.049)	0.048(0.050)	0.049(0.051)	0.051(0.049)	0.052(0.049)
AB	0.041(0.062)	0.048(0.058)	0.053(0.053)	0.059(0.048)	0.063(0.041)
PB	0.051(0.052)	0.051(0.051)	0.051(0.050)	0.050(0.052)	0.051(0.052)

Discussion

Based on these simulations, the NA and AB techniques exhibit much higher type 1 error rates for the lower-sided tests when α is small and q is big. The drawback of the NA approach decreases as grows big. Take note that the NA test becomes too cautious when both O and q are modest. All things considered, the PB test does a good job across the board in terms of type 1 mistake rates.
Then, to compare the strengths of these four tests, we do a small-scale simulation research. Without changing the size, the power comparison is done. These tests should ideally have their sizes modified to the same level so that we may compare their type 1 error rates. We will not be covering the laborious process of adjusting the size to the same level in this post.

We select $n = 10$, $\beta = 5.0$ to run the power comparison. For α and q, the choices $\alpha = 0.5$, 1.0, 1.5, 5.0 and $q = 0.1$, 0.3, 0.5, 0.7, 0.9 were selected. The power comparison is done for the lower and upper-sided hypotheses stated in (1). Furthermore, in order to display the selected choices for δ that species in (1), we introduce the ratio $c = \delta/\theta_q$. For the given values of c and q, the values of δ were selected. The simulation was done under these parameter configurations and the rejection rates are reported in Tables 3-6.

When $c = 1$, by design the null hypothesis is true and the rejection rates are the actual size (type 1 error rates) for the test. When $c>1$, the alternative hypothesis is true and the choice is associated with a lower-tailed test and the rejection rate is the power of the test for that choice. Similarly, when $c<1$, the rejection rate is the power of the upper-tailed test associated with that choice. Tables 3-6 correspond to the choices of $\alpha = 0.5$, 1.0, 1.5, 5.0, respectively.According to these limited power simulation results, the following performance patterns were observed.

For the Lower Tailed Tests

When α and q are small, both the GM and PB methods perform well. In these cases, in terms of the size, both the NA and AB tests are too conservative. When α gets larger, the size disadvantage goes away with the NA method.

When α small and q is large, the power of the NA and AB tests is higher than the GM and PB methods. However, in these cases, the NA and AB methods have highly elevated type 1 error rates. When α gets larger, the size disadvantage goes away with the NA method.

For the Upper Tailed Tests

The AB test has high power performance when p and α are modest. Nonetheless, the AB test's type 1 error rate is rather high in this instance. The AB test still has high rates of type 1 errors even when the O levels are big. The NA technique has poor performance when both O and p are small. The performance of the NA technique improves as O becomes greater. When p is big and O is small, all approaches perform similarly.
Typically, the simulations show that the NA technique performs better as the shape parameter O increases. Since a big shape parameter is predicted to lead to a less skewed gamma distribution, this is not surprising. When the gamma shape

parameter is more than 7, the gamma distribution may be approximated, the GM method and the PB method outperform them when both size and power performance are considered.

Table 3: Power performance of lower and upper-tailed tests when α = 0.5, β = 5, *n* = 10

		For lower-tailed test						For upper-tailed test			
p	*c*	NA	AB	GM	PB	*p*	*c*	NA	AB	GM	PB
.1	1.0	0.019	0.036	0.049	0.054	.1	.1	0.083	0.299	0.259	0.273
	1.5	0.042	0.073	0.090	0.098		.25	0.060	0.186	0.154	0.169
	2.0	0.072	0.115	0.136	0.156		.5	0.041	0.117	0.084	0.101
	2.5	0.106	0.157	0.191	0.201		.75	0.031	0.084	0.058	0.073
	3.0	0.145	0.203	0.232	0.251		1	0.026	0.066	0.044	0.053
.3	1.0	0.020	0.040	0.049	0.054	.3	.1	0.548	0.633	0.540	0.595
	1.5	0.075	0.123	0.173	0.158		.25	0.325	0.377	0.295	0.343
	2.0	0.166	0.240	0.307	0.286		.5	0.160	0.191	0.139	0.169
	2.5	0.280	0.367	0.436	0.416		.75	0.089	0.107	0.074	0.094
	3.0	0.404	0.492	0.558	0.540		1	0.052	0.065	0.041	0.053
.5	1.0	0.041	0.051	0.060	0.047	.5	.1	0.947	0.898	0.852	0.892
	1.5	0.152	0.175	0.221	0.168		.25	0.699	0.614	0.541	0.604
	2.0	0.317	0.353	0.406	0.351		.5	0.329	0.282	0.229	0.276
	2.5	0.493	0.535	0.594	0.530		.75	0.143	0.125	0.927	0.120
	3.0	0.649	0.691	0.732	0.685		1	0.061	0.055	0.042	0.052
.7	1.0	0.066	0.067	0.060	0.050	.7	.1	0.994	0.987	0.987	0.991
	1.5	0.220	0.220	0.210	0.170		.25	0.878	0.815	0.802	0.848
	2.0	0.420	0.420	0.405	0.346		.5	0.466	0.384	0.373	0.431
	2.5	0.612	0.609	0.580	0.524		.75	0.178	0.138	0.125	0.161
	3.0	0.757	0.758	0.733	0.677		1	0.057	0.043	0.038	0.053
.9	1.0	0.083	0.088	0.053	0.050	.9	.1	0.997	0.996	0.998	0.998
	1.5	0.249	0.254	0.188	0.152		.25	0.907	0.898	0.904	0.920
	2.0	0.448	0.449	0.332	0.291		.5	0.489	0.462	0.487	0.524
	2.5	0.632	0.623	0.479	0.442		.75	0.169	0.146	0.163	0.192
	3.0	0.768	0.755	0.607	0.568		1	0.044	0.035	0.042	0.053

able 4: Power performance of lower and upper-tailed tests when α = 1.0, β = 5, *n* = 10

		For lower-tailed test						For upper-tailed test			
p	*c*	NA	AB	GM	PB	*p*	*c*	NA	AB	GM	PB
.1	1.0	0.032	0.038	0.040	0.055	.1	10	0.349	0.594	0.491	0.547
	1.5	0.120	0.129	0.148	0.168		.25	0.211	0.360	0.237	0.316
	2.0	0.258	0.265	0.275	0.315		.5	0.112	0.189	0.088	0.153
	2.5	0.412	0.412	0.501	0.457		.75	0.067	0.110	0.041	0.085
	3.0	0.558	0.551	0.671	0.604		1	0.041	0.068	0.036	0.052
.3	1.0	0.036	0.044	0.057	0.052	.3	.1	0.906	0.937	0.899	0.915
	1.5	0.220	0.244	0.267	0.265		.25	0.639	0.693	0.600	0.651
	2.0	0.512	0.538	0.554	0.575		.5	0.294	0.340	0.258	0.296
	2.5	0.758	0.774	0.812	0.789		.75	0.125	0.151	0.108	0.124
	3.0	0.897	0.907	0.925	0.918		1	0.051	0.064	0.045	0.051
.5	1.0	0.048	0.054	0.058	0.051	.5	.1	0.998	0.998	0.996	0.998
	1.5	0.291	0.317	0.316	0.307		.25	0.927	0.916	0.884	0.909
	2.0	0.630	0.661	0.662	0.646		.5	0.518	0.514	0.442	0.496
	2.5	0.860	0.879	0.886	0.866		.75	0.186	0.190	0.150	0.173
	3.0	0.958	0.966	0.965	0.960		1	0.053	0.057	0.047	0.051
.7	1.0	0.056	0.065	0.061	0.052	.7	.1	1.000	1.000	1.000	1.000
	1.5	0.306	0.336	0.294	0.283		.25	0.987	0.982	0.983	0.988
	2.0	0.636	0.667	0.611	0.595		.5	0.684	0.655	0.647	0.684
	2.5	0.855	0.875	0.853	0.828		.75	0.240	0.222	0.209	0.162
	3.0	0.952	0.961	0.956	0.936		1	0.051	0.046	0.040	0.051
.9	1.0	0.059	0.075	0.056	0.050	.9	.1	1.000	1.000	1.000	1.000
	1.5	0.274	0.321	0.285	0.232		.25	0.992	0.990	0.989	0.993
	2.0	0.556	0.607	0.592	0.479		.5	0.721	0.659	0.662	0.736
	2.5	0.773	0.811	0.785	0.689		.75	0.248	0.221	0.182	0.270
	3.0	0.899	0.919	0.887	0.830		1	0.046	0.037	0.036	0.054

Table 5: Power performance of lower and upper-tailed tests when α = 1.5, β = 5, *n* = 10

		For lower-tailed test						For upper-tailed test			
p	*c*	NA	AB	GM	PB	*p*	*c*	NA	AB	GM	PB
.1	1.0	0.038	0.038	0.045	0.052	.1	.1	0.603	0.798	0.738	0.759
	1.5	0.204	0.194	0.223	0.246		.25	0.370	0.522	0.438	0.464
	2.0	0.461	0.441	0.466	0.499		.5	0.176	0.260	0.201	0.210
	2.5	0.692	0.672	0.707	0.712		.75	0.087	0.131	0.090	0.104
	3.0	0.846	0.831	0.849	0.854		1	0.046	0.068	0.043	0.048
.3	1.0	0.042	0.046	0.050	0.052	.3	.1	0.985	0.993	0.982	0.988
	1.5	0.349	0.365	0.399	0.383		.25	0.833	0.872	0.818	0.841
	2.0	0.747	0.759	0.770	0.769		.5	0.421	0.476	0.397	0.427
	2.5	0.939	0.942	0.950	0.947		.75	0.157	0.191	0.150	0.161
	3.0	0.990	0.990	0.992	0.991		1	0.051	0.064	0.042	0.051
.5	1.0	0.048	0.056	0.056	0.053	.5	.1	1.000	1.000	1.000	1.000
	1.5	0.411	0.437	0.443	0.417		.25	0.986	0.986	0.977	0.985
	2.0	0.816	0.837	0.843	0.824		.5	0.676	0.683	0.623	0.665
	2.5	0.969	0.975	0.977	0.970		.75	0.237	0.251	0.204	0.234
	3.0	0.996	0.997	0.994	0.997		1	0.052	0.056	0.047	0.050
.7	1.0	0.054	0.065	0.061	0.049	.7	.1	1.000	1.000	1.000	1.000
	1.5	0.392	0.433	0.405	0.371		.25	0.999	0.999	0.999	0.999
	2.0	0.781	0.813	0.791	0.761		.5	0.826	0.815	0.801	0.832
	2.5	0.950	0.961	0.950	0.942		.75	0.305	0.295	0.270	0.313
	3.0	0.991	0.994	0.994	0.988		1	0.050	0.049	0.045	0.052
.9	1.0	0.054	0.072	0.059	0.049	.9	.1	1.000	1.000	1.000	1.000
	1.5	0.321	0.381	0.322	0.296		.25	0.990	0.999	0.999	0.999
	2.0	0.655	0.717	0.653	0.610		.5	0.848	0.829	0.846	0.859
	2.5	0.867	0.902	0.861	0.828		.75	0.314	0.284	0.298	0.329
	3.0	0.957	0.971	0.952	0.933		1	0.048	0.038	0.040	0.053

Table 6: Power performance of lower and upper-tailed tests when α = 5.0, β = 5, *n* = 10

		For lower-tailed test						For upper-tailed test			
p	*c*	NA	AB	GM	PB	*p*	*c*	NA	AB	GM	PB
.1	1.0	0.048	0.037	0.050	0.055	.1	.1	0.997	1.000	0.999	0.999
	1.5	0.686	0.654	0.686	0.702		.25	0.931	0.968	0.938	0.949
	2.0	0.986	0.983	0.985	0.989		.5	0.561	0.653	0.561	0.581
	2.5	1.000	1.000	1.000	1.000		.75	0.203	0.262	0.201	0.208
	3.0	1.000	1.000	1.000	1.000		1	0.049	0.067	0.048	0.050
.3	1.0	0.049	0.049	0.057	0.054	.3	.1	1.000	1.000	1.000	1.000
	1.5	0.865	0.865	0.932	0.869		.25	1.000	1.000	1.000	1.000
	2.0	1.000	0.999	1.000	0.999		.50	0.901	0.928	0.889	0.902
	2.5	1.000	1.000	1.000	1.000		.75	0.373	0.425	0.350	0.375
	3.0	1.000	1.000	1.000	1.000		1	0.050	0.063	0.050	0.049
.5	1.0	0.050	0.057	0.058	0.051	.5	.1	1.000	1.000	1.000	1.000
	1.5	0.868	0.886	0.876	0.870		.25	1.000	1.000	1.000	1.000
	2.0	1.000	1.000	1.000	1.000		.5	0.990	0.992	0.984	0.991
	2.5	1.000	1.000	1.000	1.000		.75	0.544	0.573	0.540	0.548
	3.0	1.000	1.000	1.000	1.000		1	0.050	0.058	0.052	0.050
.7	1.0	0.050	0.063	0.053	0.051	.7	.1	1.000	1.000	1.000	1.000
	1.5	0.778	0.821	0.790	0.779		.25	1.000	1.000	1.000	1.000
	2.0	0.996	0.997	0.997	0.996		.50	0.999	0.999	0.997	0.999
	2.5	1.000	1.000	1.000	1.000		.75	0.655	0.652	0.650	0.670
	3.0	1.000	1.000	1.000	1.000		1	0.049	0.049	0.049	0.052
.9	1.0	0.051	0.069	0.053	0.049	.9	.1	1.000	1.000	1.000	1.000
	1.5	0.592	0.668	0.627	0.585		.25	1.000	1.000	1.000	1.000
	2.0	0.946	0.968	0.955	0.944		.50	0.998	0.998	0.996	0.998
	2.5	0.997	1.000	0.998	0.994		.75	0.634	0.601	0.607	0.646
	3.0	1.000	1.000	1.000	1.000		1	0.049	0.039	0.038	0.053

Conclusion

Hydrology, environmental monitoring, and life testing are just a few of the many fields that benefit greatly from statistical inference on gamma quantiles. But since the gamma distribution is so complicated, all the statistical inference methods found in books and articles are approximations. Depending on whether it is connected to the low or high quantiles, the simulation findings show that the type 1 error rates of the approximation techniques may be excessively cautious or too liberal in some instances. Two novel strategies were presented in this research; one used the Parametric Bootstrap methodology (PB), and the other the extended p-value technique (GM). These two novel methods work well over the whole quantile range of θq, where 0< q < 1. In terms of size and power performance, the GM and PB methods outperform the other two procedures regardless of whether lower or higher quantile values are involved or whether the tests are upper or lower-tailed. But since it's easier on the computer, the PB technique beats the GM method.

References

In their 2023 publication, "Communications in Statistics Theory and Methods," Ananda, M. M. A., Dag, O., and Weerahandi, S. discuss several topics including 8207-8222. Here is the link to the article: https://doi.org/10.1080/03610926.2022.2059682.
In 2000, Aksoy published a work. Integration of gamma distribution into hydrological models. Volume 24, Issue 6, Pages 419–428 of the Turkish Journal of Engineering and Environmental Sciences. Doi: 10.1080/02626669009492416 Bhaumik, S. K., Aryal, S., Mathew, T., & Gibbons, R. D. (2008). Tolerance and prediction bounds for the gamma distribution, approximately. Applied Statistical Science Journal, 16(2), 253-261. Here is the link to the article: https://doi.org/10.1080/01621459.1965.10480857. In 1998, Ashkar and Ouarda published a work. Quantiles of gamma and generalized gamma distributions with approximate confidence intervals. Publication: Journal of Hydrologic Engineering, Volume 3, Issue 1, Pages 43–51. URL: https://doi.org/10.1061/(ASCE)1084-0699(1998)3:1(43) for the citation. "Ashkar and Bobée" was published in 1988. Bayesian or log-likelihood intervals for occurrences of flooding in a JAWRA model The citation is from the American Water Resources Association's Journal, 24, 3, pages 639–650. Visit https://doi.org/10.1111/j.1752-1688.1988.tb00916.x for it. In 1984, Bain, Engelhardt, and Shiue published a paper. Bounds on the gamma distribution's approximate tolerance and confidence levels for dependability. Published in Transactions on Reliability, volume 33, issue 2, pages 184-387. You can access the article at this link: https://doi.org/10.1109/TR.1984.5221771. In 2006, Bhaumik and Gibbons published a paper. Approximate intervals for one-sided predictions for a gamma population at each of r sites, where p is less than or equal to m observations. Published in Technometrics, volume 48, issue 1, pages 112-119. DOI: https://doi.org/10.1198/004017005000000355. In 2009, Bhaumik, Kapur, and Gibbons published a work. Examining gamma distribution characteristics using limited data sets. Published in Technometrics, volume 51, issue 3, pages 326–334. You can access the article at this link: https://doi.org/10.1198/tech.2009.07038.
In 1991, Bobee and Ashkar published. We use the gamma distribution and its derivatives in hydrology.
0918334683 is the ISBN-10 number. In 1977, Engelhardt and Bain published a paper. On the scale parameter of a gamma distribution with a nuisance shape parameter, uniformly most powerful unbiased tests are used. Technical Statistics, Volume 19, Issue 1, Pages 77–81. Here is the link to the article: ://doi.org/10.1080/00401706.1977.7489502.
(Gibbons, R. D. 1995). Methods for statistically monitoring groundwater. Journal des Sciences Hydrologiques, volume 40, issue 3, pages 425. The publication's DOI is 10.1002/9780470172940. I (Guttman, 1967). Tolerance zones for statisticalanalysis.TraditionalandBayesianapproaches.thislink: tps://cir.nii.ac.jp/crid/1571698599439086976
This was published in 1986 by Hawkins and Wixley. An observation on the normalization of chi-squared variables. Publication:TheAmerican Statistician,Volume40,Issue,Pages296-298.URL: https://doi.org/10.1080/00031305.1986.10475420
Funk, C., Husak, G. J., & Michaelsen, J. (2007). For drought monitoring purposes, monthly rainfall in Africa is represented using the gamma distribution. Volume 27, Issue 7, Pages 935–944,

published by the Royal Meteorological Society's International Journal of Climatology. http://doi.org/10.1002/joc.1441 published in the Journal of Chemical Society.